Principal Bundles in Lattice Gauge Theory

Randy S

Abstract Lattice gauge theory can be used to construct many models of quantum fields that we don't know how to construct in smooth spacetime. In lattice gauge theory with gauged group G, a gauge field is described using G-valued **link variables** associated with the links in the lattice, or more generally with the edges in a graph. The graph is meant to be a kind of discrete approximation to a manifold M representing continuous space or spacetime. This article explains how that formulation of lattice gauge theory relates to principal G-bundles and connections on M.

Contents

1 Introduction

Article 70621 reviews the concept of a **principal G-bundle**, the mathematical foundation for the idea of a gauge field in physics. The bundle's base space M represents space or spacetime. The bundle is called **trivial** if its total space is $M \times G$ and its projection $M \times G \to M$ is $(m, g) \mapsto m$. Every principal bundle looks trivial locally, but many combinations of M and G admit principal bundles that are not trivial globally.

If the base space M is $\mathbb{R}^n$, then every principal bundle over M is trivial. If that were the only M we cared about, then we would have no need to consider nontrivial bundles,[1] but quantum field theory gives us reasons to consider other base spaces, too. Considering other base spaces can lead to more insight about quantum field theory,[2] and treating n-dimensional space as an n-dimensional torus is a common way to implement a long-distance regulator.[3] For these reasons and others, nontrivial principal G-bundles are important in quantum field theory.

In classical field theory, spacetime is normally treated as a smooth manifold. We would like to do that in quantum field theory, too, but in most cases we don't know how.[4] Lattice gauge theory provides nonperturbative definitions of many models that we would otherwise not know how to define, but it involves a compromise: instead of using a smooth manifold M as the base space, it uses an artificially-chosen subset $\Gamma \subset M$ to control the number of field variables. This article highlights some relationships between principal G-bundles over the base spaces M and $\Gamma \subset M$. Section 2 outlines what each section will cover.

[1] Section 2.2 in Barrett (1991) says, "Many physicists are quite happy with the idea that gauge fields are Lie-algebra-valued one-forms on the base manifold, and regard bundles as an extra complication, which one can happily do without. For them, all bundles are trivial, and the connection is a G-invariant one-form on [the base space] $\times G$."

[2] Section 2.1.2 in García-Etxebarria and Montero (2019) reviews two reasons to consider base spaces with various topologies. One of the reasons is also emphasized in `https://physics.stackexchange.com/a/659024/`.

[3] Article 51376 uses this device for the case $G = U(1)$, and article 33600 shows that nontrivial principal G-bundles over an n-dimensional torus exist for some combinations of G and n.

[4] We can do it term-by-term in an expansion in powers of a small constant ("perturbation theory"), but those expansions usually don't converge, and even if they did, that would be an unsatisfying way to define a theory. Imagine taking a first course in trigonometry where the teacher describes trigonometric functions *only* through their small-angle expansions, thoroughly obscuring one of their most important properties – their periodicity.

2 Outline

- Section 4 explains that in the context of a principal G-bundle over M, parallel transport gives a map from paths in M to elements of G only if the bundle is trivial.

- Section 5 introduces the idea that the base space in lattice gauge theory is a one-dimensional CW complex $\Gamma \subset M$ (called a **graph** in this article,[5] also called a **lattice** in much of the physics literature) and reviews a traditional interpretation of the link variables in lattice gauge theory in terms of parallel transport, which only makes sense when the bundle is trivial.

- Section 7 uses transition functions to show that if G is connected, then every principal G-bundle over a graph $\Gamma \subset M$ is trivial. Section 8 uses a property of the classifying space BG to show (again) that if G is connected, then every principal G-bundle over a graph $\Gamma \subset M$ is trivial. Section 9 reiterates the conditions under which this conclusion applies.

- To reinforce the conclusion, section 10 shows that any nontrivial principal G-bundle over a torus M can be made trivial by removing just a small part of M that doesn't intersect $\Gamma \subset M$.

- Section 11 shows that when G is connected, any configuration of the (classical) gauge field on a graph $\Gamma \subset M$ can be realized by a connection on a trivial principal G-bundle defined over all of M.

- Sections 12-13 explain how lattice gauge theory manages to encode information about nontrivial principal G-bundles in a continuum limit, even when Γ itself admits only trivial principal G-bundles. Sections 14-19 work through an example.

- Section 20 explains what happens when the gauged group G is not connected.

[5] The first page of chapter 10 in May (2007) says, "A graph is a one-dimensional CW complex."

3 Conventions and notation

- G is a Lie group.
- M is a manifold used as the base space of a principal G-bundle.
- A one-dimensional CW complex is called a **graph**.
- Γ is a graph (the **lattice** in lattice gauge theory) whose sites and links are regarded as points and line segments in M.
- ℓ is a link in Γ, and $\square$ is a plaquette.[6]
- A **configuration of the (lattice) gauge field** is an assignment of an element of G, denoted $g(\ell)$, to each link ℓ.
- ϵ is the smallest nonzero distance between sites in Γ.
- X is a generic base space, not necessarily a manifold or a graph.
- A is a local potential – a representation of the gauge field as a one-form defined on part of M.
- F is the field strength, represented as a 2-form on M.
- ω is a connection, represented as a one-form on the total space.[7]
- σ is a local section (a smooth function from part of M into G).
- S^n is an n-dimensional sphere, so S^1 is a circle.
- BG is a classifying space for G.

[6] A **plaquette** is a loop formed by four links.

[7] Article 70621 defines the **total space** of a fiber bundle.

4 Parallel transport in a principal G-bundle

Article 76708 reviews the concept of a **connection** on a principal G-bundle and the related concept of **parallel transport**. This section explains why parallel transport along a path γ in the base space M can be described as multiplication by an element $g(\gamma)$ of the gauged group G only if the bundle is trivial.

A principal G-bundle over a base space M is called **trivial** if is isomorphic to one whose total space is $M \times G$ and whose bundle projection is the obvious projection from $M \times G$ to M.[8] Other bundles are called **nontrivial**.

If a principal G-bundle over M is trivial, then the fiber over each point $x \in M$ may be regarded as a copy of the gauged group G. In that case, parallel transport from one fiber to another along a given path γ in M may be described as multiplication by a group element $g(\gamma) \in G$.

In a nontrivial principal G-bundle, that doesn't make sense. The fibers cannot be regarded as copies of the gauged group G in a way that varies smoothly throughout all of M.[9] The action of the gauged group G on each individual fiber is defined, so parallel transport around a *closed* loop γ can be described as multiplication by a (γ-dependent) element of G, but if we want to express parallel transport along a non-closed path γ as multiplication by an element of G, then we need to choose reference points in each of the fibers being compared – we need to choose a **local section** – so that each point in a given fiber may be obtained by acting on its reference point with a specific element of G.[10] Nontrivial principal bundles don't have continuous global sections,[11] so we can't choose those reference points in a way that varies smoothly across all of the fibers in the bundle.

[8] Section 1

[9] The fibers may still be regarded as copies of $\tilde{G}$, where $\tilde{G}$ is almost the same as G but without any distinguished identity element (article 70621).

[10] When we represent a connection using a one-form A on part of the base space (A is called a **local potential**), like traditional presentations of quantum field theory always do, we have implicitly chosen a local section: the local potential A is the pullback of the connection one-form by a local section (article 76708).

[11] Article 70621

5 Parallel transport in lattice gauge theory

When the base space M is a smooth manifold, parallel transport can be defined along any curve γ in M. In lattice gauge theory, parallel transport is defined only along a discrete set of paths, namely those that belong to a one-dimensional **CW complex**[12] $\Gamma \subset M$. The structure Γ consists of a discrete set of **0-cells** (which will be called **sites**) and a discrete set of **1-cells** (which will be called **links**). Each site is a point in M, and each link is a path in M that connects a pair of sites to each other. If we treat the graph as a mere combinatorial structure by thinking of each link as nothing more than a pair of points (the link's endpoints),[13] then Γ reduces to what mathematicians call a **graph**. The topological structure will be important in this article, but the name *graph* will still be used because it's shorter than "one-dimensional CW complex."

In conventional lattice gauge theory with gauged group G and graph $\Gamma \subset M$, one G-valued **link variable** is assigned to each link γ in Γ. The value of a link variable is often interpreted as describing parallel transport along that link. As explained in section 4, this interpretation make sense everywhere on Γ only if the principal G-bundle on M becomes trivial when restricted to the graph $\Gamma \subset M$. Sections 7-8 will show that if the gauged group G is connected, then every principal G-bundle over a graph Γ is indeed trivial, so the usual interpretation of link variables as implementing parallel transport is justified.

Section 20 will explain how this interpretation should be generalized when the gauged group G is not connected.

[12] Article 93875 reviews the definition of *CW complex*.

[13] This ignores the topological structure, which treats each 1-cell as a continuous 1-dimensional entity connecting its two endpoints.

6 Continuum limits in lattice gauge theory

If we only consider observables whose resolution is much coarser than the length of any link in Γ, then we might intuitively expect a model's predictions to depend only on the manifold M and not on the details of Γ, as long as Γ samples M faithfully enough. This is roughly what *continuum limit* means in the context of lattice gauge theory. Sections 12-13 will explain how lattice gauge theory manages to encode information about nontrivial principal G-bundles in a continuum limit, even when Γ itself admits only trivial principal G-bundles. Sections 14-19 will work through an example. The examples shows that a given configuration of the gauge field on $\Gamma \subset M$ may be consistent with two or more non-isomorphic bundles on M, but no more than one of them has a field strength whose magnitude has a sensible value in the continuum limit. That one bundle is not necessarily the trivial one.

7 Principal G-bundles over a graph

Article 33600 shows that when G is connected, every principal G-bundle over a graph (one-dimensional CW complex) is trivial. This section outlines another way to reach the same conclusion. Section 8 will outline yet another way to reach the same conclusion again.

Any principal G-bundle over a base space X may be assembled from trivial bundles $U_j \times G$, where $U_1, U_2, ...$ is a collection of contractible open subsets of X whose union is X, using **transition functions** $U_j \cap U_k \to G$ to specify how these trivial bundles are glued together. Article 70621 reviews this construction in more detail. The important thing to know here is that the resulting bundle over X is trivial if each transition function is homotopic to the constant function that maps its whole domain to the identity element of G.[14]

Now suppose that X is a graph that doesn't have any individual links from a site to itself.[15] Then X can be covered with contractible open sets $U_1, U_2, ...$, each containing exactly one site and covering the nearest two-thirds of each link that touches that site. Each intersection $U_j \cap U_k$ is a union of disjoint contractible regions – a union of middle-thirds of links. If G is connected, then any function from such a region into G is homotopic to the constant function that maps the whole region to the identity element of G, so the resulting bundle over X is trivial.

If the graph is a subset $\Gamma \subset M$ of an n-dimensional manifold with $n \geq 2$, then the reasoning and the conclusion still hold even if Γ is "thickened" to an n-dimensional manifold $\hat{\Gamma}$ formed by the union of tiny n-dimensional neighborhoods of each point of Γ, as long as the radius of each neighborhood is much less than the distance between sites so that Γ is a deformation retract of $\hat{\Gamma}$.[16,17,18]

[14] Article 33600

[15] If X does have such links, then we can insert a site in the middle of each of them to get a topologically equivalent graph that doesn't have any such links. The graphs normally used in lattice gauge theory don't have any such links.

[16] Article 61813 defines **deformation retract**.

[17] If Γ is pictured as a network of line segments, then $\hat{\Gamma}$ may be pictured as the same network drawn with a slightly thicker pencil, as long as the pencil's tip is still much narrower than the distance between neighboring sites.

[18] If Γ were a submanifold of M, then the definition of $\hat{\Gamma}$ as a "thickened" version of Γ would be essentially the concept of a **tubular neighborhood** of Γ (Kosinski (1993), chapter 3, definition 2.4).

8 An approach using the classifying space BG

Let $[X, Y]$ denote the set of homotopy classes of maps from X into Y,[19] and let BG be a **classifying space** for G, a topological space with this property:[20] for each CW complex X, elements of $[X, BG]$ are in 1-to-1 correspondence with isomorphism classes of principal G-bundles over X. At least one principal G-bundle over X always exists, namely the trivial bundle. This corresponds to the fact that at least one homotopy class of maps $X \to BG$ always exists, namely the class containing a constant map (one that maps X to a single point of BG). If every continuous map $X \to BG$ is homotopic to a constant map, so that $[X, BG]$ has only one element, then every principal G-bundle over X is trivial.

The homotopy groups of a classifying space BG are related to those of the group G by[20]

$$\pi_n(BG) \simeq \pi_{n-1}(G). \tag{1}$$

The $n = 1$ case of this relationship says that if the group G is connected ($\pi_0(G) = 0$), then the classifying space BG is simply-connected ($\pi_1(BG) = 0$).[21]

If X and X' are homotopy equivalent to each other, then $[X, BG] = [X', BG]$.[22] We can use this to simplify the determination of $[X, BG]$ when X is a connected graph. Every connected graph X has a **spanning tree** (also called a **maximal tree**),[23] a connected sub-graph that includes all of X's 0-cells but doesn't have any cycles: every pair of 0-cells is connected by exactly one path made of 1-cells. Any tree is contractible when treated a topological space (a one-dimensional CW complex), so X is homotopy equivalent to the quotient space X/T_X defined by collapsing a spanning tree T_X to a single point.[24] The tree T_X contains every 0-cell of X, so X/T_X consists of a single 0-cell with some number of 1-cells attached to it at both ends, one for each 1-cell in X that isn't in T_X. In other words, X/T_X is a

[19] Article 69958

[20] Article 35490

[21] Cohen (2023), corollary 4.10

[22] Cohen (2023), theorem 4.1 and corollary 4.3

[23] Collins (2018), definition 6.1

[24] Collins (2018), proposition 6.3

wedge sum of circles, denoted $S^1 \vee S^1 \vee \cdots \vee S^1$.[25] Altogether, the original graph X is homotopy equivalent to a wedge sum of circles,[26] so

$$[X, BG] = [X/T_X, BG] = [S^1 \vee S^1 \vee \cdots \vee S^1, BG]. \tag{2}$$

Now we can use this relationship:[27]

$$[S^1 \vee S^1 \vee \cdots \vee S^1, BG] = [S^1, BG] \times [S^1, BG] \times \cdots \times [S^1, BG], \tag{3}$$

where $\times$ is the usual cartesian product of sets. The $n = 1$ case of equation (1) says that if G is conneccted, then $[S^1, BG]$ has only one element, and then equations (2) and (3) imply that $[X, BG]$ also has only one element, so every principal G-bundle over a graph is trivial when G is connected.

If the graph is a subset $\Gamma \subset M$ of an n-dimensional manifold with $n \geq 2$, then the "thickened" graph $\hat{\Gamma}$ that was defined in section 7 is homotopy equivalent to Γ, so $[\hat{\Gamma}, BG] = [\Gamma, BG]$. This shows again that replacing Γ with $\hat{\Gamma}$ doesn't change the conclusion.

[25] Article 69958 defines *wedge sum*.

[26] Barz (2023), last sentence in the proof of corollary 5.5; May (2007), chapter 4, first theorem in section 3

[27] Article 69958

9 The scope of the conclusion

The conclusion in sections 7-8 relies on certain conditions: the base space is a graph, G is connected, and the bundle is a *principal* G-bundle. This section emphasizes that the conclusion can be evaded when any of those conditions is not satisfied.

Article 33600 constructs examples of nontrivial principal G-bundles over some n-dimensional base spaces with $n \geq 2$ for connected Lie groups G, so the conclusion of sections 7-8 can be evaded when the base space is not a graph.

If the group G is not connected, then the argument in section 7 doesn't apply because even if the domain X of a function $X \to G$ is contractible, the function might still fail to be homotopy equivalent to the constant function that maps all of X to the identity element of G. This is the case whenever the function's image includes an element of G that is not connected to the identity element by any continuous path in G. The reasoning in section 8 also doesn't apply in this case, because if G is not connected, then $\pi_0(G)$ has more than one element, and therefore so does $\pi_1(BG)$.[28] As a result, nontrivial principal G-bundles may exist over a graph when G is not connected. Section 20 will say more about this.

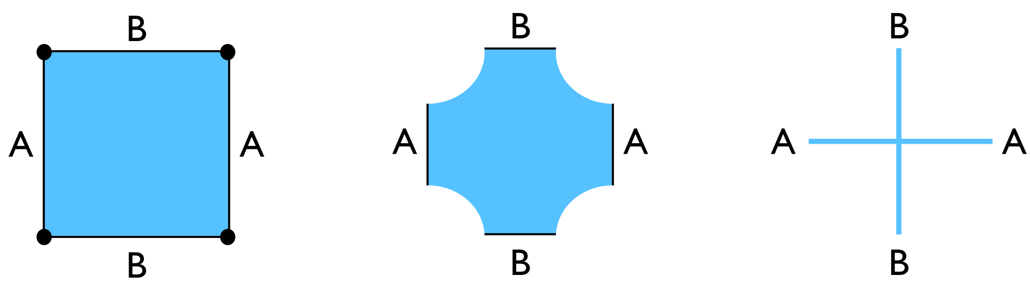
Finally, the reasoning in sections 7-8 is only for *principal* G-bundles, not for other fiber bundles whose fiber happens to be homeomorphic to G. The reasoning in section 7 used this condition implicitly by assuming that the transition function can be implemented as multiplication by a G-valued function. Example: the Klein bottle is the total space of a nontrivial bundle whose base space is a circle (which can be viewed as a graph with two sites and two links) and whose fiber is homeomorphic to $U(1)$, but it's not a *principal* $U(1)$-bundle because it uses a transition function involving complex conjugation, which isn't an element of $U(1)$. The reasoning in section 8 is also specific to *principal* G-bundles through its dependence on the classifying space BG.

[28]In particular, if G is discrete (which implies not connected), then the **Eilenberg-MacLane space** $K(G, 1)$ is a classifying space for G, and its fundamental group is nontrivial (isomorphic to G itself), so nontrivial G-bundles may exist over a graph when G is discrete (article 35490 and section 20).

10 How fragile is a bundle's nontriviality?

Sections 7-8 showed that when G is connected, all principal G-bundles over a graph are trivial and that the conclusion still holds when the graph $\Gamma \subset M$ is "thickened." This section provides another perspective on that result by answering this question: starting with a nontrivial principal G-bundle over a torus $M = S^1 \times \cdots \times S^1$, with a connected Lie group G,[29] how much of M must we delete to ensure that the remaining principal G-bundle is trivial?

Article 33600 shows that when the base space is a two-dimensional torus ($M = S^1 \times S^1$) deleting a single point is sufficient. Here's the argument again, re-arranged to allow a more concise generalization to higher-dimensional tori. In the series of pictures shown below, the first picture represents the torus as a square with opposite sides identified. The four corners all represent a single point in the torus. The second and third pictures illustrate the fact that if we delete this one point (represented by the four corners), then the remaining manifold – called a **once-punctured torus** – can be deformation-retracted to a pair of circles that intersect each other at a point.



This shows that a punctured torus is homotopy equivalent to a wedge sum of two circles, so if G is connected, then all principal G-bundles over a punctured torus are trivial.[30] Given a lattice $\Gamma \subset S^1 \times S^1$, we can remove any one point from $S^1 \times S^1$

[29] Article 33600 shows that nontrivial principal $U(1)$-bundles exist over an n-dimensional torus for every $n \geq 2$.

[30] Footnote 37 in section 14 will use Stokes's theorem $\int_M dA = \int_{\partial M} A$ to show that a principal $U(1)$-bundle over a two-dimensional torus cannot be trivial if the integral of the field strength two-form over the whole torus is nonzero. Removing a single point from the domain of integration doesn't change the value of the integral, but it does invalidate

that doesn't belong to Γ, so this shows again that every principal G-bundle over a two-dimensional lattice (used as a discrete approximation to a torus) is trivial when G is connected.

For an n-dimensional torus with $n \geq 2$, deleting a single point is not enough,[31] but the preceding construction still has a natural generalization to arbitrary n:

- A three-dimensional torus ($n = 3$) may be represented as a cube with opposite faces identified. Removing the cube's corners and edges (which form a network of codimension-2 manifolds) leaves a manifold that is homotopy equivalent to a wedge sum of three circles, so all principal G-bundles over that manifold are trivial when G is connected.

- More generally, an n-dimensional torus may be represented as an n-cube with opposite $(n-1)$-dimensional faces identified. Removing the $(n-2)$-dimensional faces of those $(n-1)$-dimensional faces leaves a manifold that is homotopy equivalent to a wedge sum of n circles, so all principal G-bundles over that manifold are trivial when G is connected.

Given a lattice Γ as a subset of an n-dimensional torus M, we can choose the deleted parts of M in the preceding construction so that none of them intersect Γ,[32] so every principal G-bundle over an n-dimensional lattice (used as a discrete approximation to an n-dimensional torus) is trivial when G is connected.

that application of Stokes's theorem. Stokes's theorem assumes that A has compact support (Madsen and Tornehave (1997), theorem 10.8), which is automatically true if the manifold M is compact (like a torus), but it's not necessarily true if M is non-compact (like a punctured torus).

[31] Deleting a single point from an n-dimensional torus leaves something homotopy equivalent to n intersecting $(n-1)$-dimensional tori. (Intuition: think of the n-dimensional torus as an n-dimensional hypercube with opposite $(n-1)$-dimensional faces identified, and delete the point represented by the hypercube's 2^n corners. Eat into the corners until the remainder is squeezed as much as possible.)

[32] Choose an integer $K \gg 1$. Define the torus to be the subset of $\mathbb{R}^n$ in which each coordinate is restricted to the range $[-(K+1/2),\ (K+1/2)]$, with opposite sides identified. Define a lattice to consist of all points with integer coordinates. Delete all points for which at least two coordinates have magnitude $K+1/2$. The remainder includes all sites and links of the lattice, because they each have no more than one coordinate with magnitude $K+1/2$. (Each link is a line segment along which only one coordinate varies, and the values of the other coordinates are integers.)

11 Compatible connection on a trivial bundle

This section shows that when G is connected, every configuration of the gauge field on a graph $\Gamma \subset M$ is consistent with a principal connection defined on the trivial principal G-bundle, over the whole manifold M.

A connection on a trivial bundle over M is determined by a local potential on M,[33] so the result may be established by constructing a local potential one-form A on M. The idea is to construct a one-form A that is zero everywhere except in a tiny neighborhood of the midpoint of each link $\ell \in \Gamma$, where A is chosen to reproduce the value of the link variable $g(\ell) \in G$ through the formula

$$g(\ell) = P \exp\left(i \int_\ell A\right), \tag{4}$$

where P stands for **path-ordered**.[34] In more detail:

- For each ℓ, let $A(\ell)$ be an element of the Lie algebra satisfying $g(\ell) = \exp(iA(\ell))$.
- Let $f(\ell, x)$ be a function of $x \in M$ whose integral along link ℓ is 1 and whose support is limited to a small neighborhood of the midpoint of that link so that $f(\ell, x)f(\ell', x) = 0$ if $\ell' \neq \ell$.
- Take the local potential one-form to be $A = \sum_\ell f(\ell, x)A(\ell)dx_\ell$, where the sum is over only one direction of each undirected link and where the coordinate x_ℓ is the only one that's not constant on link ℓ.

The fact that such a one-form A can be constructed shows that when G is connected, the link variables on a graph $\Gamma \subset M$ may always be interpreted in terms of parallel transport defined by a connection on the manifold M, as promised in section 5.

[33] If the bundle is trivial, then any Lie-algebra-valued one-form on M determines a connection (Figueroa-O'Farrill (2010), section 5.2.4).

[34] Article 76708

12 The magnitude of the field strength

Section 11 showed that when G is connected, a configuration of the gauge field on a graph $\Gamma \subset M$ may always be interpreted in terms of parallel transport defined by a connection on the manifold M. Even though such a connection always exists, it is not always the best way to extend a configuration of the link variables on Γ to a connection on M. This section explains why.

When a quantum field model is defined by treating space as a lattice with distance ϵ between adjacent sites, only states whose energy remains finite in the continuum limit (loosely described as the limit $\epsilon \to 0$) can represent physically meaningful quantities in the corresponding quantum field theory in continuous spacetime.[35] That's really just the definition of the continuum limit: in quantum field theory, treating space as a lattice is a compromise used to achieve mathematical clarity, and only predictions that are restricted to resolutions much coarser than the lattice scale ϵ can be directly relevant to physics. The next paragraph shows that the field strength of the connection that was constructed in section 11 typically diverges in the limit $\epsilon \to 0$. The quantum model's hamiltonian (energy operator) includes terms proportional to the square of the field strength, so the connection in section 11 is typically not an appropriate continuous-space interpretation of a given configuration of the link variables in the context of any physically relevant state.

As an example, consider the case $G = U(1)$, represented as the group of complex numbers z satisfying $|z| = 1$ with ordinary multiplication as the group operation. If some of the link variables $g(\ell)$ differ significantly from the identity element $1 \in U(1)$, then equation (4) implies that the components A_k of the local potential one-form $A = \sum_k A_k(x)dx^k$ have magnitude $\sim 1/\epsilon$, because the integral is over a link ℓ of length ϵ. For the same reason, the function $f(\ell, x)$ defined below that equation must have slope $\sim 1/\epsilon$ somewhere along each link ℓ, so the derivatives $\partial_j A_k$ have magnitude $\sim 1/\epsilon^2$. The components F_{jk} of the field strength tensor are given by $F_{jk} = \partial_j A_k - \partial_k A_j +$ (nonlinear terms), so F_{jk} has magnitude $\sim 1/\epsilon^2$.

[35] Article 10142

13 Nontrivial bundles and the continuum limit

The connection constructed in section 11 is not the only connection on M consistent with the specified values $g(\ell)$ of the link variables. The same configuration of the lattice gauge field is also consistent with other connections M, including some for which the principal G-bundle is trivial and possibly some for which it is not. Only a connection whose field strength is small compared to $1/\epsilon^2$ can provide an appropriate continuous-space interpretation of a given configuration of the lattice gauge field. Among principal G-bundles on M and connections that satisfy this constraint, a given configuration of the lattice gauge field might not be consistent with any of the trivial-bundle options but might be consistent with a nontrivial-bundle option instead. In this sense, lattice gauge theory defined on a graph $\Gamma \subset M$ can emulate features of nontrivial principal G-bundles over M even if every principal G-bundle on Γ is trivial, which is the case when G is connected. Examples:

- Phillips (1984) explains how to construct a principal $U(1)$-bundle and connection on an n-dimensional torus corresponding to a given configuration of the lattice gauge field when the flux through each plaquette is not too large.[36] The bundle is not necessarily trivial.

- Lüscher (1982) and Phillips and Stone (1986) do a similar thing for the case $G = SU(2)$ when the graph approximates a four-dimensional base space.

Sections 14-19 will work through an example with $G = U(1)$ when the lattice Γ approximates a two-dimensional torus.

[36] A **plaquette** is a loop formed by four links. In that analysis, "not too large" means that the magnitude of the flux through each plaquette must be less than π. As emphasized in article 51376, the flux in a $U(1)$ model is defined only modulo 2π anyway, so this constraint really only eliminates one possible value of the flux (modulo 2π), namely π.

14 Introduction to an example

As explained in section 13, the same configuration of the gauge field on $\Gamma \subset M$ may be consistent with different connections on different principal G-bundles on M, but only a connection whose field strength is small compared to $1/\epsilon^2$ is appropriate as a continuous-space interpretation of that lattice configuration. Sections 15-19 will illustrate that situation using $G = U(1)$ as the group, using $M = S^1 \times S^1$ as the base space, and using a fine-mesh lattice as the graph $\Gamma \subset M$. Outline:

- Section 15 will define the lattice Γ and describe a configuration of the gauge field on Γ that is consistent with a field strength of small constant magnitude everywhere on M.

- Section 16 will show that if a connection on a trivial bundle reproduces that configuration of the lattice gauge field, then it necessarily has a large ($\sim 1/\epsilon^2$) field strength somewhere.

- Section 17 will construct a principal $U(1)$-bundle over M and a connection whose field strength has small constant magnitude everywhere on M. The bundle is necessarily nontrivial, because the integral of the field strength over M is nonzero.[37]

- Section 18 will construct a local section that is defined almost everywhere on M, excluding only a tiny region that doesn't intersect any links of the lattice.

- Section 19 will use that local section to construct a local potential that reproduces the gauge-field configuration in section 15.

This illustrates the idea that a given configuration of the gauge field on a lattice $\Gamma \subset M$ is consistent with no more than one principal bundle on M if the field strength of the connection is required to have magnitude $\ll 1/\epsilon^2$ everywhere, and the bundle selected by this constraint may be nontrivial.

[37] If the field strength 2-form F could be written $F = dA$ for a local potential one-form A defined on all of M, then Stokes's theorem would give $\int_M F = 0$. If $\int_M F \neq 0$, then F cannot be written $F = dA$ for any local potential A defined on all of M, which in turn implies that the bundle must be nontrivial.

15 Example, part 1: a constant-flux configuration

The base space in this example is a 2-dimensional torus, $M = S^1 \times S^1$. Represent the torus as $\mathbb{R}^2$ modulo the equivalence relation

$$(x+1, y) \sim (x, y) \qquad\qquad (x, y+1) \sim (x, y) \tag{5}$$

for all $(x, y) \in \mathbb{R}^2$. Choose an integer $N \gg 1$, define $\epsilon \equiv 1/N$, and take the sites of the lattice to be the points of the form

$$(x, y) = (n_x\epsilon,\ n_y\epsilon)$$

for integers n_x, n_y. Take the links to be the segments that connect each site (x, y) to the neighboring sites $(x \pm \epsilon, y)$ and $(x, y \pm \epsilon)$, and remember the equivalance relation (5).

Assign a value $g(\ell) \in G$ to each directed link, and let $g(\Box_{\text{cc}})$ denote the product of those values in the counterclockwise direction around the perimeter of a plaquette. Section 17 will construct a principal $U(1)$-bundle over M and a connection whose field strength 2-form has constant magnitude everywhere on M. For the quantities $g(\ell)$ to be consistent with that connection, the holonomies $g(\Box_{\text{cc}})$ must have the same value for every plaquette. Let z denote that value. The number of plaquettes is N^2, and the product of all of the holonomies $g(\Box_{\text{cc}})$ must be 1 (because the link variables cancel each other pairwise in that product), so the value z must satisfy $z^{(N^2)} = 1$. This implies $z = e^{2\pi ik/N^2}$ for some integer k.

Figure 1 shows one way to choose the link-variable values $g(\ell)$ so that $g(\Box_{\text{cc}})$ is equal to z for the counterclockwise holonomy around each plaquette (equivalently, equal to $1/z$ for every clockwise holonomy). The quantity w in that figure is defined by $w \equiv 1/z^N$. Each line segment is a link, and the arrows specify the orientation of the link to which the indicated power of z or w is assigned. The lower-left corner of the grid is the point $(x, y) = (0, 0)$.

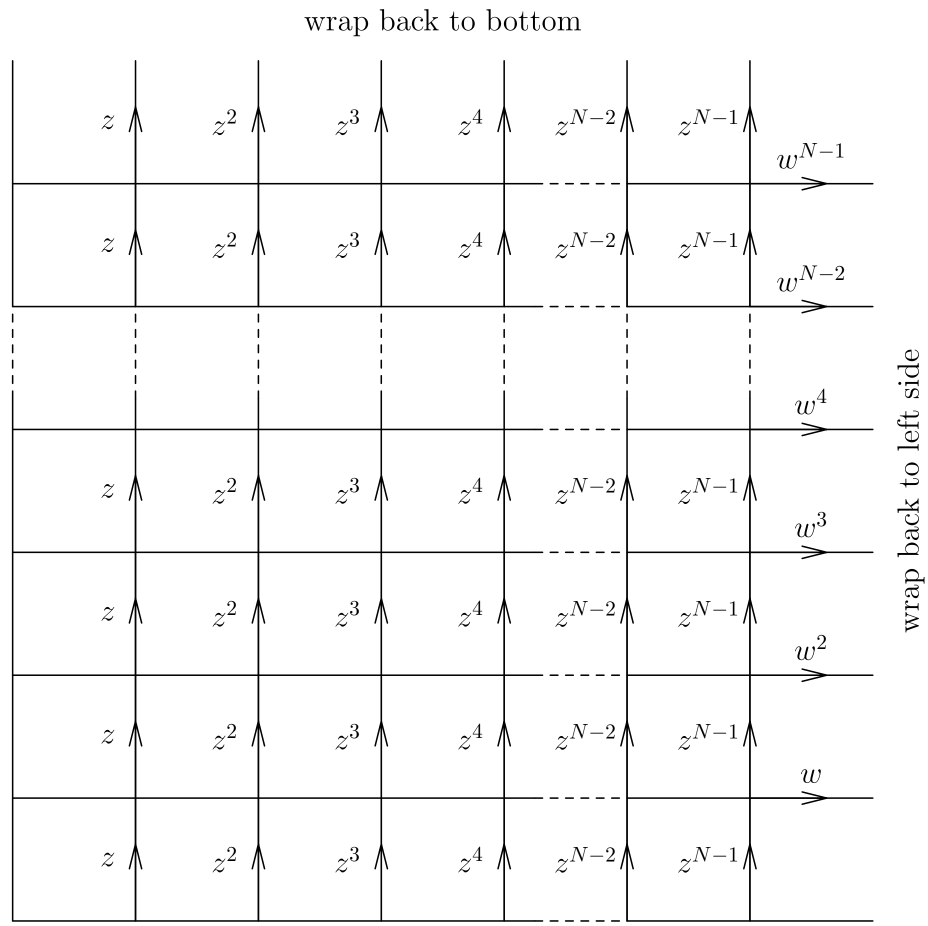


Figure 1 – Configuration of a $U(1)$ gauge field on a lattice version of a two-dimensional torus, with N steps per dimension. *Configuration* means that a specific element of $U(1)$ is assigned to each directed link. The value assigned to unlabeled links is 1. The value assigned to labeled links is either a power of z or a power of w, with the direction indicated by the arrow. Reversing the direction of the arrow corresponds to replacing the assigned value with its inverse. If $z^{(N^2)} = 1$ and $w = 1/z^N$, then the product of link variables taken counterclockwise around each plaquette is equal to z. In that case, this configuration is consistent with a nontrivial principal $U(1)$-bundle over the underlying continuous two-dimensional torus, with a connection whose field strength has a constant nonzero magnitude everywhere. This same configuration of the link variables is also consistent with a trivial principal $U(1)$-bundle over the underlying continuous two-dimensional torus, as any configuration of the link variables must be, but the field strength of the corresponding connection on that trivial bundle must have a large magnitude somewhere (diverging as $N \to \infty$), so the nontrivial-bundle "interpretion" of the lattice configuration is more natural than the trivial-bundle "interpretation."

16 Example, part 2: the trivial bundle

Section 11 already showed that any configuration of the $U(1)$ lattice gauge field on $\Gamma \subset M$ is consistent with a connection on the trivial principal $U(1)$-bundle on M. This section shows that if a connection on the trivial principal $U(1)$-bundle on M is consistent with the configuration described in section 15, then the the magnitude of the field strength is necessarily large ($\sim 1/\epsilon^2$) somewhere on M. For this reason, the trivial-bundle interpretation of that configuration is not the appropriate one in the continuum limit.[38]

Suppose that A is a local potential one-form that reproduces the values of all of the link variables:

$$g(\ell) = \exp\left(i\int_\ell A\right). \tag{6}$$

The product $g(\square_{\text{cc}})$ of the link variables around the perimeter of a plaquette is

$$g(\square_{\text{cc}}) = \exp\left(i\int_{\partial\square} A\right) \tag{7}$$

where $\int_{\partial\square}$ denotes an integral around the perimeter of the plaquette in the counterclockwise direction. As a subset of M, the plaquette is a contractible region, so Stokes's theorem gives

$$\int_{\partial\square} A = \int_\square F,$$

where $F \equiv dA$ is the field strength 2-form and $\int_\square$ denotes an integral over the area of the plaquette. The configuration described in section 15 has the property

$$g(\square_{\text{cc}}) = e^{2\pi ik/N^2} \tag{8}$$

for every plaquette. Here, as in section 15, the group $U(1)$ will be represented as the group of complex numbers with magnitude 1 using multiplication as the group operation. We can also represent $U(1)$ as as $\mathbb{R}$ modulo 2π with addition as

[38] Sections 12-13

the group operation.[39] The quantities (6) and (7) both take values in $U(1)$. In contrast, the quantities A and F both take values in the Lie algebra, which is $\mathbb{R}$ without the modulo-2π relation. For this reason, equations (7)-(8) don't determine $\int_\square F$ uniquely. The only imply

$$\int_\square F = \frac{2\pi k}{N^2} + 2\pi n_\square \tag{9}$$

with undetermined integers $n_\square$. These integers are determined when we choose a specific bundle and connection on M that is compatible with the configuration on $\Gamma \subset M$.

The lattice has N^2 plaquettes, so the total flux over all of M is

$$\int_M F = \sum_\square \int_\square F = 2\pi k + 2\pi \sum_\square n_\square. \tag{10}$$

The area of a plaquette is $\epsilon^2 = 1/N^2$, so the average field strength in any given plaquette is

$$\frac{1}{\epsilon^2}\int_\square F = 2\pi k + \frac{2\pi n_\square}{\epsilon^2}, \tag{11}$$

Now, suppose the bundle is trivial. In this case, we can write $F = dA$ for a one-form A that is defined on all of M, and then Stokes's theorem gives $\int_M F = \int_{\partial M} dA$. Since M is a compact manifold without a boundary (∂M is empty), this implies $\int_M F = 0$. Use this in equation (10) to deduce that if $k \neq 0$, then at least one of the integers $n_\square$ must also be nonzero, and then equation (11) implies that the magnitude of F is $\sim 1/\epsilon^2$ somewhere. Altogether, this shows that if the bundle is trivial, then the field strength cannot have magnitude $\ll 1/\epsilon^2$ everywhere on M unless $k = 0$. For this reason, if $k \neq 0$, then the trivial-bundle interpretation of that configuration is not the appropriate one in the continuum limit.

Conversely, if $k \neq 0$ and the field strength has magnitude $\ll 1/\epsilon^2$ everywhere on M, then the bundle must be nontrivial. Sections 17-19 will construct such a bundle and a connection for which all of the integers $n_\square$ in (9) are zero.

[39] Sections 17-19 will use this representation.

17 Example, part 3: a compatible nontrivial bundle

To construct the total space of the bundle, represent $U(1)$ as $\mathbb{R}$ modulo 2π. Use coordinates (x, y, θ) to represent a point in $\mathbb{R}^2 \times U(1)$. Let k be the integer that was denoted by the same symbol in section 15, and define the quotient space $E \equiv (\mathbb{R}^2 \times U(1))/\sim$ where $\sim$ is the equivalence relation defined by

$$(x+1, y, \theta) \sim (x, y, \theta + 2\pi k\, y) \qquad\qquad (x, y+1, \theta) \sim (x, y, \theta). \tag{12}$$

for all $(x, y) \in \mathbb{R}^2$ and $z \in U(1)$ Then E is the total space of a principal $U(1)$-bundle over the base space $M \equiv S^1 \times S^1$ with projection $p : (x, y, \theta) \mapsto (x, y)$.

Choose the connection[40]

$$\omega = d\theta + 2\pi k\, x\, dy. \tag{13}$$

This is consistent with the equivalence relations (12), because

$$d\theta + 2\pi k\,(x+1)\,dy = d(\theta + 2\pi k\, y) + 2\pi k\, x\, dy.$$

With this connection, the vector fields ∂_x and $\partial_y - 2\pi k\, x\partial_\theta$ are both horizontal.[41]

The curvature 2-form $d\omega$ is defined everywhere on the total space E. The field strength F is a 2-form on the base space. To calculate it, choose a covering $U_1, U_2, ...$ of the base space M by contractible charts, and choose local sections $\sigma_k : U_k \to E$. The field strength in U_k is defined as the pullback of $d\omega$ by σ_k.[42] In symbols: $F|_{U_k} = \sigma_k^* d\omega$. Use (13) to get $d\omega = 2\pi k\, dx \wedge dy$. The pullback σ_k^* has no effect in this case because $dx \wedge dy$ is already a 2-form on the base space, so $F|_{U_j} = F|_{U_k}$ in $U_j \cap U_k$, as anticipated by the notation $F|_\bullet$. As a result, the field strength 2-form

$$F = 2\pi k\, dx \wedge dy \tag{14}$$

is defined on the whole base space M. The integral of F over the whole base space is $\int_M F = 2\pi k$, so the bundle is nontrivial if $k \neq 0$.

[40] For convenience, the connection is taken to be real-valued in this example (instead of Lie-algebra valued, which would introduce an overall factor of $-i$).

[41] With respect to a given connection one-form ω, a vector field v is called **horizontal** if $\omega(v) = 0$ (article 76708).

[42] Article 76708

18 Example, part 4: a section, almost everywhere

A local potential depends both on the connection ω and on a choice of local section σ. A local section is a smooth map from part of the base space M to the group $U(1)$. We can't define a section everywhere on M because the bundle is nontrivial,[43,44] but we can define one almost everywhere on M – everywhere except a single point that we are free to choose.

To construct such a local section, start with the non-continuous function σ' from M to E defined by

$$\sigma'(x,y) = \begin{cases} (x,y,0) & \text{for } 0 \leq x < 1-\epsilon/2, \\ (x,y,-2\pi k\, y) & \text{for } 1-\epsilon/2 \leq x \leq 1. \end{cases} \tag{15}$$

This satisfies $\sigma'(1,y) = \sigma'(0,y)$, thanks to the equivalence relation (12). For each y in the range $0 < y < 1$, this is a continuous function of x except for a discontinuity at $x = 1-\epsilon/2$. This discontinuity occurs at the midpoint of each of the links labeled with a power of w in figure 1. Clearly, (15) is a continuous function of x for $y = 0$, and it is also continuous for $y = 1$ because θ and $\theta + 2\pi$ are equivalent to each other in $U(1)$. By modifying $\sigma'(x,y)$ in a tiny neighborhood of the discontinuity, we can smooth out the discontinuity for all but one value of y in the range $0 < y < 1$. Choose $y = 1-\epsilon/2$ to be the one value of y at which the discontinuity at $x = 1-\epsilon/2$ remains, and let $\sigma(x,y)$ denote the resulting function. Then $\sigma(x,y)$ is a local section, defined everywhere on M except the single point

$$(x,y) = (1-\epsilon/2,\ 1-\epsilon/2). \tag{16}$$

In particular, it is defined on all of the sites and links of the lattice $\Gamma \subset M$.

Section 19 will use σ to construct a local potential that reproduces the configuration of the lattice gauge field shown in figure 1.

[43] Article 70621

[44] If σ were not required to be continuous, then of course we could define it everywhere on M. The requirement for σ to be continuous is what prevents us from defining it everywhere on M. In this article, σ is also required to be smooth (so the resulting local potential is differentiable), but that doesn't impose any additional restrictions on its domain of definition.

19 Example, part 5: a compatible local potential

The connection ω is a one-form on the total space E. A local potential A is corresponding a one-form on part of the base space M. This section constructs a local potential A that is defined along all of the links in the lattice $\Gamma \subset M$ and shows that it reproduces the configuration of the lattice gauge field depicted in figure 1.

Let σ be the local section that was defined in section 18, and let M_σ be the part of M on which σ is defined. The manifold M_σ is a punctured torus, the result of removing a single point (16) from the torus M. Define the local potential[45]

$$A = \sigma^*\omega. \tag{17}$$

The connection ω is defined everywhere in the total space E, and the local section σ is defined on M_σ, so the local potential A is also defined everywhere on the punctured torus M_σ.

The holonomy around the perimeter of a plaquette $\square$ is[45]

$$\exp\left(i\int_\square F\right). \tag{18}$$

A plaquette has area $\epsilon^2 = 1/N^2$, so the quantity (18) is equal to the quantity z that was defined in section 15. In fact, the local potential (17) reproduces the specific configuration of the gauge field that was depicted in figure 1. To confirm this, recall[45] the meaning of the pullback σ^* in equation (17): it means that the result of evaluating[46] the one-form A on any vector field v on M_σ is

$$A(v) \equiv \omega(\sigma_* v), \tag{19}$$

where $\sigma_* v$ is the pushforward of v to a vector field on the total space E. If we think of v as the derivative along a curve $\gamma \subset M_\sigma$, then the pushforward $\sigma_* v$ is

[45] Article 76708

[46] Recall (article 09894) that a one-form is a linear map from vector fields to scalar fields.

the derivative along the curve $\sigma(\gamma) \subset E$. We can use this to derive an explicit expression for A from equation (13) for ω.

First consider the region $M_0 \subset M_\sigma$ consisting of points (x, y) with $0 \leq x \leq 1-\epsilon$. In that region, $\sigma(x, y) = (x, y, 0)$, so if a vector v in M_σ is represented by a linear combination of the partial derivatives ∂_x and ∂_y, then the pushforward $\sigma_* v$ is still represented by that same linear combination of partial derivatives, without a ∂_θ term. This implies that in M_0, the local potential one-form A is obtained from ω by omitting the $d\theta$ term on the right-hand side of (13):[47]

$$A = 2\pi k\, x\, dy \qquad\qquad (\text{in } M_0).$$

The region M_0 corresponds to the part of figure 1 that remains after omitting the right-most column of horizontal links (the ones labeled by powers of w). Within this region, the integral of A along any horizontal link (constant y) is zero, and the integral of A along any vertical link (constant x) is $2\pi k x \epsilon$. This shows that the relationship

$$g(\ell) = \exp\left(i \int_\ell A\right) \tag{20}$$

reproduces the configuration in figure 1 throughout the region M_0.

Now consider the region $M_1 \subset M_\sigma$ consisting of points (x, y) with $1-\epsilon \leq x \leq 1$, not including the point (16). In figure 1, M_1 contains the right-most column of horizontal links (the ones labeled by powers of w). In that region, $\sigma(x, y)$ is a smoothed version of the function (15). Along any given horizontal (constant-y) link in that region, the function $\sigma(x, y)$ has the form $\sigma(x, y) = (x, y, \theta(x))$, where $\theta(x)$ is a smooth function that starts at $\theta(1-\epsilon) = 0$ and ends at $\theta(1) = -2\pi k\, y$. The connection ω doesn't have a dx term, so the integral of A along that link is determined entirely by the $d\theta$ term, with the result $\int_\ell A = \int_\ell \sigma^*\omega = \int_\ell d\theta = -2\pi k\, y$. Using this in (20) gives the gauge-field configuration (the part involving powers of w) shown in figure 1.

[47] This gives $dA = 2\pi k\, dx \wedge dy$ in M_0, which is consistent with equation (14) because $F = dA$.

20 What if G is not connected?

Now suppose that the group G is not connected, and suppose that the graph Γ is not a tree.[48] Use equation (1) together with the fact that $\pi_0(G)$ has more than one element to deduce that $\pi_1(BG)$ also has more than one element. Then use equations (2)-(3) to deduce that $[\Gamma, BG]$ has more than one element, which says that nontrivial principal G-bundles over Γ exist.

Sections 4-5 explained that if the bundle over Γ is trivial, then the individual link variables in lattice gauge theory can be interpreted in terms of parallel transport. What if the bundle is not trivial? The previous paragraph showed that this question matters when G is not connected.

Even if the bundle is nontrivial, parallel transport around a closed path (loop) is still represented by an element of G. Within the category of smooth bundles over closed manifolds M, this map from the set of loops in the base space M into G is enough information to uniquely determine both the bundle and the connection (up to equivalence).[49] A given map from loops in M into G might not come from any connection on any principal G-bundle over M, but if it does, then it determines both of them uniquely.

Similarly, we can think of a configuration of a lattice gauge field as specifying both a principal bundle over the graph Γ and parallel transport along its links.[50] If G is connected, then the bundle is trivial, so the link variables *only* describe parallel transport. If G is discrete, then parallel transport is trivial,[51] so the link variables *only* specify the bundle.[52] If G is neither discrete nor connected, like $G = O(n)$ with $n \geq 2$, then the link variables do both.

[48] The graphs (lattices) used in lattice gauge theory are never trees, because a plaquette is a closed loop.

[49] Barrett (1991); Lewandowski (1993), theorem 4.4; Caetano and Picken (1994)

[50] The relationship is many-to-one, because different configurations may specify the same bundle and connection if the configurations are related to each other by a gauge transformation.

[51] When G is discrete, a given principal G-bundle admits only one connection.

[52] We can think of the graph Γ as a collection of open sets that each contains just one site and part of each link that touches that site, and then (when G is discrete) we can think of the link variables as transition functions used to glue those patches together (article 33600).

21 Significance for quantum field theory

The perspective highlighted in section 20 has this significance for quantum models with gauge fields:

- In the path-integral formulation of lattice gauge theory, the sum over configurations of the gauge field may be interpreted as a sum over principal G-bundles together with a sum over connections on those bundles.[53]

- In the hamiltonian formulation of lattice gauge theory, the construction of the Hilbert space accounts for all of the possible principal G-bundles and all of the possible connections on them. A typical state is a quantum superposition of many different configurations, typically involving different bundles as well as different connections.

As explained in sections 12-13, through an appropriately-defined continuum limit, the path-integral sum and the Hilbert space implicitly account for any nontrivial principal G-bundles over a corresponding smooth manifold M even though the configurations are defined on a lattice $\Gamma \subset M$ that doesn't admit any nontrivial principal G-bundles when G is connected.

[53]Separating those two types of information from each other (the isomorphism class of the bundle and the connection) might not be easy, but it's also not necessary, because lattice gauge theory sums over both of them anyway.

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