Event Horizons Without Coordinate Singularities: the Schwarzschild Metric in Kerr-Schild Coordinates

Randy S

Abstract This article introduces the spacetime geometry induced by an isolated non-rotating spherical body in general relativity. This is a good approximation to the geometry of spacetime near the earth, and it also applies to a non-rotating black hole. This article uses some easy calculations and flat-spacetime analogies to help develop intuition about the event horizon, emphasizing that it is locally nothing special even though it plays a special role on larger scales.

Contents

1 Introduction

Section 4 introduces the Schwarzschild metric, which serves as a relatively simple example of a curved spacetime metric that satisfies the Einstein field equation (the equation of motion for the metric field in general relativity) for a semi-realistic configuration of matter. According to general relativity, the Schwarzschild metric describes the space-time outside any isolated central body that is spherically symmetric, electrically neutral, and non-rotating.[1] It is a good approximation near the earth. The Schwarzschild metric depends only on the mass of the central body and is independent of the body's radius, so no extra effort is required to entertain questions about what would happen if the body's mass were compressed enough to form a black hole.[2] Real black holes are expected to be rotating very rapidly, so the line element introduced in this note is not quite appropriate for realistic black holes, but it still illustrates some important concepts.

The metric depends only on the object's mass M, not on the object's size, but it is often written in terms of a constant R called the **Schwarzschild radius**[3]

$$R = 2GM \tag{1}$$

where G is Newton's gravitational constant. Even though it's called a radius and has units of length, R is determined by the object's *mass*, not by the object's *size*. Examples:

- If M is the mass of the earth, then R is rougly 9 millimeters.
- If M is the mass of the sun, then R is rougly 3 kilometers.

[1] This is a loose statement of a consequence of Birkhoff's theorem (page 125 in section 6.1 of Wald (1984)). Birkhoff's original theorem is specific to four-dimensional spacetime. Regarding generalizations to higher-dimensional spacetime, see section 1.5.1 in Chapter 1 of Horowitz (2012).

[2] For a more thorough online introduction to the mathematics of black holes, I recommend Townsend (1997).

[3] This article uses units in which the speed of light c is equal to one. In units where $c \neq 1$, equation (1) becomes $R = 2GM/c^2$.

2 Review of the metric concept

Article 48968 is a prerequisite for this one. Here's a quick review.

In a given coordinate system, a metric may be specified in a compact way by specifying the **line element**

$$\sum_{a,b} g_{ab}(x)\,dx^a\,dx^b. \tag{2}$$

The coefficients $g_{ab}(x)$ are the **components** of the metric in the given coordinate system. In this expression, x is a point in spacetime with coordinates x^a. The sign of the line element determines whether a infinitesimal displacement $(dx^0, dx^1, ...)$ is timelike or spacelike, and then the square root of its absolute value gives the displacement's proper duration or proper length, respectively. In these expressions, each superscript is an index, not an exponent.

For the rest of this article, the coordinates will be denoted (w, x, y, z) instead of (x^0, x^1, x^2, x^3).

Example: in *flat* spacetime, we can choose a coordinate system in which the the line element is

$$dw^2 - (dx^2 + dy^2 + dz^2).$$

Now each superscript is an exponent, not an index. The notation dw^2 is an abbreviation for $(dw)^2$, and likewise for the other coordinates. With this line element, the equation for the proper time τ along any timelike worldline is

$$d\tau^2 = dw^2 - (dx^2 + dy^2 + dz^2). \tag{3}$$

The equation for the proper distance along any spacelike worldline is similar but with the opposite overall sign. This article is mainly concerned with the motion of test objects, whose worldlines are necessarily timelike (or lightlike if massless), so the metric is most conveniently specified by writing the equation for proper time.

3 How to make spherical symmetry evident

The line element of a non-rotating black hole is a simple modification of (3). It does not preserve the x, y, z-translation symmetry of (3), but it does preserve spherical symmetry, so I'll use a notation that helps make the spherical symmetry evident. Despite their name, so-called spherical coordinates *obscure* the symmetry instead of making it evident. They are convenient for some things, but not for highlighting spherical symmetry! For that reason, this article does not use spherical coordinates.

Spherical symmetry is more evident when the line element (3) is written as

$$d\tau^2 = dw^2 - d\mathbf{x} \cdot d\mathbf{x} \tag{4}$$

with $\mathbf{x} = (x, y, z)$ and

$$\mathbf{A} \cdot \mathbf{B} \equiv \sum_n A_n B_n.$$

The abbreviation

$$r \equiv \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{5}$$

is also useful. Here, we should think of r as a function of the coordinates $\mathbf{x}$, not as an independent coordinate. The independent coordinates are still w and $\mathbf{x}$. The identity

$$dr = \frac{\mathbf{x} \cdot d\mathbf{x}}{r} \tag{6}$$

is easy to derive by taking the differential of $r^2 = \mathbf{x} \cdot \mathbf{x}$.

The Schwarzschild metric will be written in terms of dw, dr, and $d\mathbf{x} \cdot d\mathbf{x}$, with coefficients that depend only on r. This spherical symmetry clear, because all of these quantities are clearly invariant under any linear transformation $\mathbf{x} \to \mathbf{x}'$ for which $\mathbf{x}' \cdot \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}$. In other words, they're all clearly invariant under rotations.

4 The metric

The Schwarzschild metric is defined implicitly by this line element:

$$d\tau^2 = dw^2 - d\mathbf{x}\cdot d\mathbf{x} - \frac{R}{r}(dw+dr)^2. \tag{7}$$

The independent coordinates are w and $\mathbf{x} = (x, y, z)$, and r is the function defined by (5), so dr is given by (6). A timelike worldline is one for which the right-hand side of (7) is positive, in which case τ is the proper time along that worldline. A lightlike worldline is one for which the right-hand side of (7) is zero. Section 6 will specify the **time orientation**, which says which of the two directions along each timelike worldline goes toward the future.

Equation (7) describes the Schwarzschild metric in a special coordinate system called **Kerr-Schild** coordinates.[4] The Schwarzschild metric is traditionally written in a different coordinate system. The traditional form is derived from (7) in section 16. In that section, we will see why starting with equation (7) is better.

For all $r > 0$, the metric (7) satisfies the Einstein field equation for *empty* spacetime, so it is only relevant outside of the massive central body.[5] In that context, the **singularity** at $r = 0$, where metric is undefined, is not relevant. It becomes relevant if the mass of the central body is concentrated inside $r < R$, thanks to theorems about the unavoidability of singularities under such circumstances,[6] but those theorems should be kept in perspective. They are based on general relativity, and we know that general relativity is only an approximation, because it ignores quantum effects. The occurrence of a singularity is a symptom that we have pushed this approximation too far. In the real world, in situations where general relativity predicts a singularity, *something* very interesting must happen – but general relativity itself can't tell us what that something will be.

[4] Write $w = v - r$ to get the metric in **ingoing Eddington-Finkelstein** coordinates $v, \mathbf{x}$.

[5] Inside the central body, the metric is different.

[6] This subject is reviewed in Witten (2019).

5 The signature

The spacetime metric is supposed to have lorentzian signature (article 48968). To confirm that (7) has lorentzian signature, we can use two steps:

- Confirm that the determinant of the metric is nonzero everywhere. The signature cannot vary in spacetime unless the determinant goes through zero somewhere, so this shows that the signature is the same everywhere.

- The metric obviously has lorentzian signature in the limit $r \to \infty$, where the line element reduces to $dw^2 - d\mathbf{x} \cdot d\mathbf{x}$. We know from the first step that the signature is the same everywhere, so this shows that the signature is lorentzian everywhere.

Now we just need to confirm that the determinant of the metric is nonzero everywhere. The components of the metric depend only on r, so its determinant must depend only on r. For this reason, we don't lose any generality by only considering points with $y = z = 0$ and $x > 0$. Then $dr = dx$, so the line element (7) reduces to

$$d\tau^2 = dw^2 - dx^2 - dy^2 - dz^2 - \frac{R}{x}(dw + dx)^2 \tag{8}$$

at that point. Comparing this to (2) shows that the non-zero components g_{ab} of the metric at this point are

$$\begin{pmatrix} g_{ww} & g_{wx} & & \\ g_{xw} & g_{xx} & & \\ & & g_{yy} & \\ & & & g_{zz} \end{pmatrix} = \begin{pmatrix} 1-V & -V & & \\ -V & -(1+V) & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

with $V \equiv R/x$. The determinant of this matrix is

$$(V-1)(V+1) - V^2 = -1.$$

This is nonzero for all w and r, so the determinant is the nonzero everywhere. This completes the proof that the geometry defined by (7) has lorentzian signature.

6 The time orientation

By definition, a timelike worldline is one for which the right-hand side of (7) is positive, but we still need to specify the **time orientation**, which says which of the two directions along a timelike worldline goes toward the future. The orientation used in this article is:

For $r > R$, the future is the direction of increasing w.

This implicitly fixes the time orientation everywhere, because if a timelike worldline goes from $r > R$ to $r < R$, then w continues to increase along the worldline when $r < R$, even though the w-direction is spacelike for $r < R$.[7] To see this, set $dw = 0$ in equation (7) to see that if w stops increasing anywhere along a worldline, then the worldline must be spacelike there. Therefore, $dw/d\tau$ cannot change sign along any timelike worldline, where τ is the worldline's proper time.

[7] More precisely, the vector field $\partial/\partial w$ is spacelike for $r < R$. And by the way, in this coordinate system, the vector field $\partial/\partial r$ is spacelike for all r.

7 The event horizon

With the choice of time orientation highlighted in section 6, the metric (7) describes a black hole with an **event horizon** at $r = R$: objects[8] that start with $r < R$ cannot ever reach $r > R$.

For radial motion, the proof is easy. Consider a causal (timelike or lightlike) worldline parameterized by λ, with

$$x(\lambda) > 0 \qquad\qquad y(\lambda) = 0 \qquad\qquad z(\lambda) = 0.$$

Any radial worldline can be put into this form by rotating the x, y, z coordinates. For such a worldline, the proper-time equation (7) implies

$$\begin{aligned}\dot{\tau}^2 &= \dot{w}^2 - \dot{x}^2 - \frac{R}{x}(\dot{w} + \dot{x})^2 \\ &= (\dot{w} + \dot{x})\left(\dot{w} - \dot{x} - \frac{R}{x}(\dot{w} + \dot{x})\right)\end{aligned} \tag{9}$$

where an overhead dot means a derivative with respect to λ. Now suppose that the worldline includes an event with $x = R$. At this event, the preceding equation simplifies even more:

$$\dot{\tau}^2 = -2(\dot{w} + \dot{x})\dot{x}. \tag{10}$$

The right-hand side is non-negative because we're considering a causal worldline. We can choose the parameter λ to be increasing into the future, which implies $\dot{w} > 0$ with the time orientation specified in section 6. But then the right-hand side of (10) cannot be positive unless $\dot{x} < 0$, so if a timelike worldline passes through an event with $x = R$, then the motion must be ingoing (toward smaller x). Similarly, the right-hand side of (10) cannot be zero unless $\dot{x} \leq 0$, so if a lightlike worldline passes through an event with $x = R$, then it cannot be outgoing. Altogether, this shows that a radially-moving object that starts with $r < R$ cannot ever reach $r > R$.

[8] Here, I'm using the word "object" even if the worldline is lightlike.

8 Lightlike radial worldlines and the horizon

The previous section focused on the behavior of causal radial worldlines at an event for which $r = R$. This section considers lightlike radial worldlines at all events with $r > 0$.

By definition, a lightlike worldline has $\dot{\tau} = 0$, so for lightlike radial worldlines, equation (9) reduces to

$$\text{either} \qquad \dot{w} + \dot{x} = 0 \qquad \text{or} \qquad \dot{w} - \dot{x} = \frac{R}{x}(\dot{w} + \dot{x}). \tag{11}$$

Using the identity $\dot{x}/\dot{w} = dx/dw$, the first case implies $dx/dw = -1$, so this motion is ingoing for all $x > 0$ (with the time-orientation chosen in section 6). The second case implies

$$\frac{dx}{dw} = \frac{x - R}{x + R}, \tag{12}$$

which is outgoing for $x > R$ and ingoing for $0 < x < R$. Altogether:

- At an event with $r > R$, a lightlike radial motion can be ingoing or outgoing.
- At an event with $r < R$, a lightlike radial motion can only be ingoing.

This is consistent with the result derived in section 7.

9 Timelike radial worldlines and the horizon

Start with equation (9) again, this time applied to a timelike radial worldline. For a timelike worldline, the right-hand side must be positive, so either both factors are positive, or both factors are negative. After a little re-arranging, these two cases are

$$\begin{aligned} dx/dw &< (x-R)/(x+R) \quad \text{if } dx/dw > -1 \\ dx/dw &> (x-R)/(x+R) \quad \text{if } dx/dw < -1. \end{aligned}$$

The second line contradicts itself because $(x-R)/(x+R)$ cannot have a magnitude greater than 1, so we are left with the first line, which can also be written like this:

$$-1 < \frac{dx}{dw} < \frac{x-R}{x+R}.$$

When $x < R$, this says that a timelike worldline must be ingoing: the motion must be in the direction of decreasing x, with the orientation chosen in section 6. This is consistent with the result derived in section 7.

10 Circular worldlines and the horizon

The previous sections considered only radial motion. This section shows that a circular worldline with $r < R$ cannot be causal (timelike or lightlike).

Choose two constants A, B and consider the circular worldline

$$(w, x, y, z) = (\lambda,\ A\cos B\lambda,\ A\sin B\lambda,\ 0).$$

This implies $r = A$, so $dr = 0$. Substitute this into (7) and use $d\lambda = dw$ to get

$$d\tau^2 = (1 - (AB)^2)dw^2 - \frac{R}{|A|}dw^2.$$

The right-hand side is negative if $|A| < R$, so the worldline cannot be causal (timelike or lightlike) unless $|A| > R$.

Most causal wordlines do not correspond to objects in free-fall. Objects in free-fall are described by special worldlines called **geodesics**. A circular causal geodesic describes an object in a circular *orbit*. Such geodesics are possible only if $|A| \geq 3R/2$. This result is derived in article 33547.

11 An event-horizon analog in flat spacetime

The preceding sections emphasized that an object that starts at $r < R$ cannot ever reach $r > R$. That might seem to suggest that $r = R$ is a "special place" in the Schwarzschild spacetime, but we need to be careful. It is special globally, but not locally. Locally, a sufficiently small neighborhood of any event with $r = R$ is practically indistinguishable from flat spacetime. Section 13 proves this by direct calculation.

To complement that direct calculation, this section shows that features which might seem to locally characterize the event horizon are also present in flat spacetime. Specifically:

1. We'll define a coordinate r for which $r = R$ represents the (flat spacetime analog of an) event horizon.

2. We'll consider an object hovering at a fixed "distance" from $r = R$, in the sense that if the object stops hovering and starts free-falling, then the (proper) time it takes to reach $r = R$ is the same no matter when it started falling.

3. An object hovering at a fixed "distance" from $r = R$ (as defined above) cannot receive signals from the other side of $r = R$.

4. If an object hovering at a fixed "distance" from $r = R$ drops a beacon, then the object sees light from the beacon becoming increasingly weaker and increasingly redshifted as the beacon approaches $r = R$. The hovering object never sees the beacon reach $r = R$.

The details of each item in this list are given below.

Start with the line element of flat spacetime in the standard coordinate system,

$$\dot{\tau}^2 = \dot{w}^2 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \tag{13}$$

and define $r \equiv R + x - w$, so that the condition $r = R$ is equivalent to $x = w$. This lightlike hypersurface will serve as our flat-spacetime analog of an event horizon.[9] This is item 1 in the list.

For item 2, define new coordinates ρ, ϕ implicitly by

$$w = \rho \sinh \phi \qquad\qquad x = \rho \cosh \phi. \tag{14}$$

Substitute this into (13) to get[10]

$$\dot{\tau}^2 = \rho^2 \dot{\phi}^2 - (\dot{\rho}^2 + \dot{y}^2 + \dot{z}^2). \tag{15}$$

This ρ, ϕ, y, z coordinate system doesn't cover the whole spacetime (it only covers the part with $w^2 > x^2$), but within the part it covers, we can use (15) and (13) interchangeably. For fixed $\rho_0 > 0$, consider a worldline of the form[11]

$$\phi = \lambda \qquad\qquad \rho(\lambda) = \rho_0 \qquad\qquad y(\lambda) = 0 \qquad\qquad z(\lambda) = 0, \tag{16}$$

as shown in figure 1. This worldline is timelike, and it represents an object hovering at a fixed "distance" from the horizon in the sense defined above. To prove this, use the fact that the line element (15) is invariant under ϕ-translations, so the proper duration of the falling object's journey to the horizon cannot depend on when it starts falling. To see that this duration is finite, suppose that the object starts falling at $\phi = 0$ (event B in figure 1). This point on the object's worldline corresponds to $(w, x) = (0, \rho_0)$, and its derivatives with respect to $\lambda = \phi$ are $(\dot{w}, \dot{x}) = (\rho_0, 0)$. A worldline representing free-fall is a geodesic (article 33547), and the geodesic with those conditions is[12]

$$w(\lambda) = \lambda \qquad\qquad x(\lambda) = \rho_0. \tag{17}$$

Use this in (13) to see that the proper duration of the falling object's journey to the horizon $w = x$ is $\Delta\tau = \rho_0$, which is finite. Altogether, this satisfies item 2 in the list.

[9] In this context, the hypersurface $x = w$ is called a **Rindler horizon**.

[10] This is called **Rindler coordinates** (article 48968).

[11] The worldlines in equation (16), (17), and (18) are parameterized independently of each other. The parameters are all denoted λ, but that's not meant to suggest any relationship between them.

[12] I won't bother writing the y, z-coordinates anymore, because they're zero on the worldlines that we're considering.

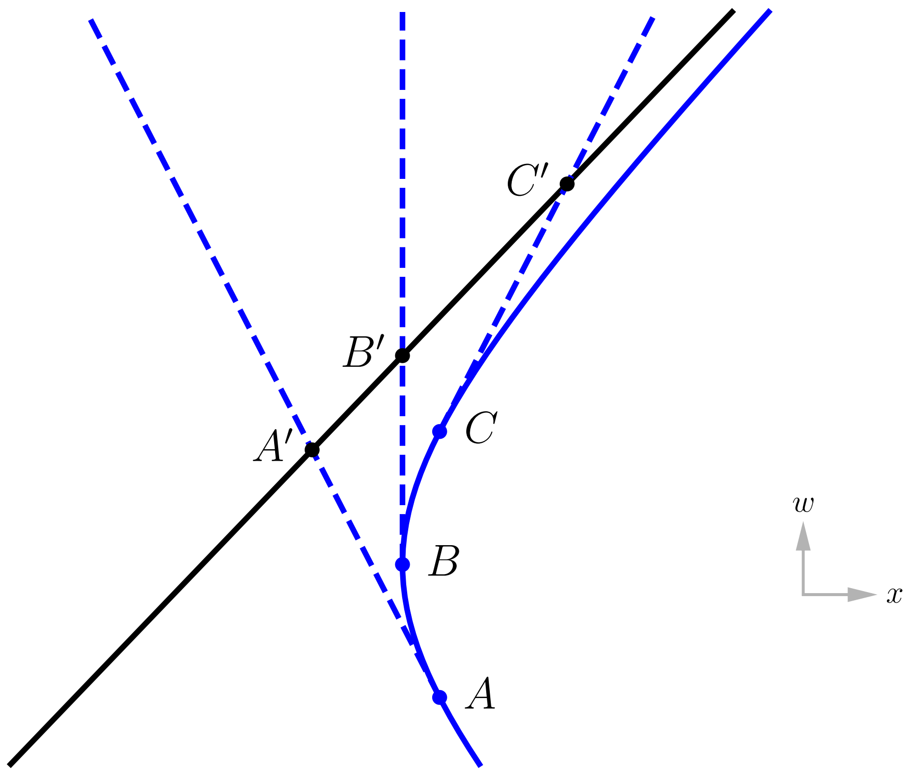

Figure 1 – The diagonal black line is the lightlike hypersurface $w = x$, which plays the role of an "event horizon" in this comparison. The y, z coordinates are suppressed in this picture. (To include them, we could use a movie instead of a static picture, with the w coordinate playing the role of "time" in the movie. In that movie, the hypersurface $w = x$ is an infinite y-z plane moving in the $+x$ direction at the speed of light.) The solid blue arc is the timelike worldline (16), which describes an object with constant absolute acceleration – equivalently, with constant weight (article 33547). We can think of the accelerating object as "hovering" outside the horizon. The object's (absolute) acceleration prevents it from ever crossing the horizon – equivalently, it prevents the horizon from ever catching up to the object. The dashed line emanating from event A is the worldline that the object would follow if it started free-falling (stopped accelerating in the absolute sense) at that event, in which case it would cross the horizon at event A'. The other dashed lines similarly show the worldlines the object would follow if it started free-falling (stopped accelerating in the absolute sense) at event B or C. Equation (13) can be used to show that the timelike segments A-A' and B-B' and C-C' all have the same proper duration. In this sense, the accelerating object is hovering at a fixed "distance" outside the horizon. Remember (article 48968) that the picture cannot faithfully convey the geometry. The picture only conveys the *coordinates* of events and worldlines. The *geometry* (proper durations and proper lengths) is defined by equation (13).

For item 3, use the fact that every event along the worldline (16) has $x > w$. An event with $x > w$ cannot be reached by any signal that starts with $x \leq w$, because a worldline connecting those two events would need to have $\dot{x} > \dot{w}$ somewhere, which makes it spacelike according to equation (13). Altogether, this shows that the object (16) cannot receive signals from events on the other side of the horizon $w = x$. This satisfies item 3 in the list.

For item 4, suppose that the hovering object drops a beacon at $\phi = 0$, so the beacon's worldline is given by (17). The analysis for item 3 already showed that the hovering object never sees the beacon reach the horizon $w = x$, and this is also clear from figure 2. But what *does* the hovering object see? To study what the hovering object sees, consider a lightlike worldline from the beacon to the hovering object. Such a worldline has the form

$$w(\lambda) = \lambda + \epsilon \qquad\qquad x(\lambda) = \lambda + \rho_0 \tag{18}$$

for some constant ϵ. According to equations (17) and (13), this worldline leaves the beacon when the beacon's proper time is ϵ. This worldline (18) intersects the worldline (16) of the hovering object whenever their w- and x-coordinates are equal. In particular, they have the same value of $x - w$ at the intersection. Use equations (14) to see that this condition is

$$\rho_0 - \epsilon = \rho_0 \cosh\phi - \rho_0 \sinh\phi,$$

which can also be written

$$\epsilon = \rho_0 - \rho_0 e^{-\phi}. \tag{19}$$

Equation (15) shows that ϕ is proportional to the hovering object's proper time ($\tau = \rho_0\phi$), so given a time on the hovering object's clock at which a signal from the beacon is received, equation (19) tells the time on the beacon's clock at which that same signal was emitted. Equation (19) has these properties:

- $\phi = 0$ corresponds to $\epsilon = 0$,
- ϵ is an increasing function of ϕ,

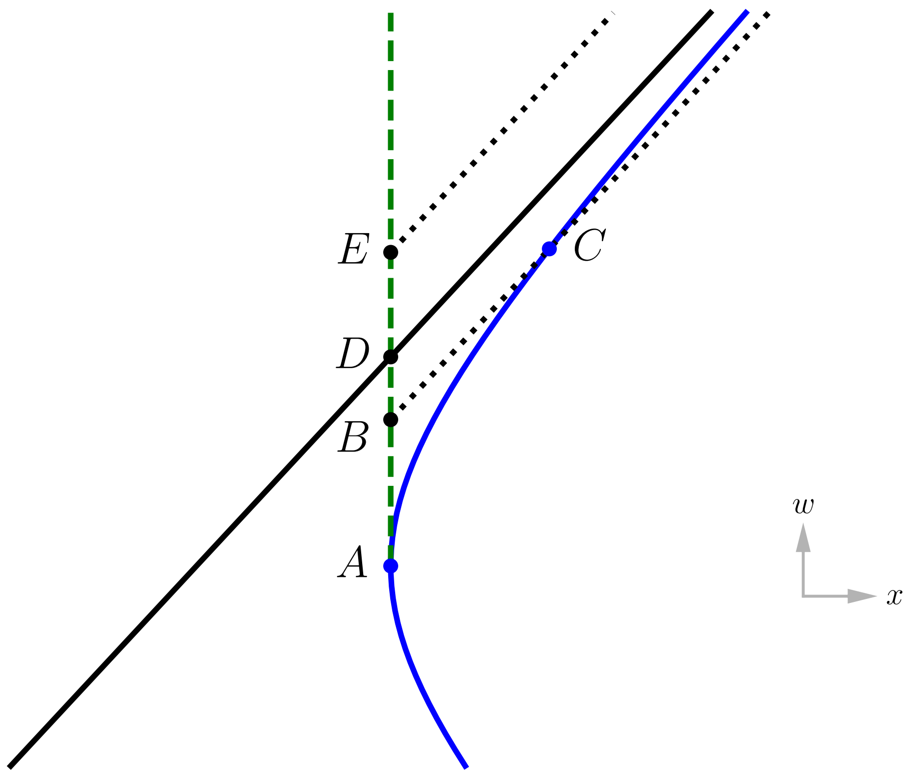

Figure 2 – The solid black diagonal line and the solid blue arc are the same as in figure 1. The vertical green dashed line is the worldline of a free-falling beacon that is released by the accelerating object at event A. The dotted diagonal line emanating from event B is an infinitesimal part of a signal (light) that the beacon emits as it falls. That part of the beacon's signal is "seen" by the accelerating object at event C. The beacon crosses the horizon at event D. As long as the object maintains the same constant absolute acceleration, the part of the signal emitted by the beacon at that event is never seen by the accelerating object – which is the same as saying that the accelerating object never crosses the horizon. Parts of the beacon's signal emitted later, such as at event E, are also never seen by the accelerating object.

- $d\epsilon/d\phi$ is a decreasing function of ϕ,
- The limit $\phi \to \infty$ corresponds to $\epsilon \to \rho_0$ and $d\epsilon/d\phi \to 0$.

The third property says that the hovering object sees light from the beacon become increasingly redshifted as the beacon approaches the horizon $w = x$, and the fourth property says that the degree of redshift diverges as the beacon approaches the horizon $w = x$, as we might have anticipated from the fact that the hovering object cannot receive signals from the beacon after the beacon crosses the horizon $w = x$. These same properties of equation (19) also imply that the hovering object sees the light from the beacon becoming increasingly weaker: if the beacon is emitting energy at a constant rate on its own clock, then the energy is being received at an increasingly-reduced rate on the hovering object's clock. Altogether, this satisfies item 4 on the list.

12 An application of the flat-spacetime analog

The analysis in the previous section shows that features that might seem to locally characterize the event horizon of a black hole actually don't, because those same local features occur even in flat spacetime. Flat spacetime is invariant under translations in every direction, so the hypersurface $w = x$ that played the role of the "horizon" in this analysis is obviously *not* a special place – even though it has the same local features that we associate with the event horizon of a black hole. The horizon is a lightlike hypersurface, and its local properties can be understood by thinking about a lightlike hypersurface in flat spacetime.

As an application, consider this common question: If a hovering observer releases a beacon, letting it fall toward the event horizon, does the hovering observer see the beacon *freeze* at the horizon? Or does the hovering observer see the object *disappear*? In the flat-spacetime analogy described above, as the beacon approaches the horizon, the beacon's apparent motion slows for the same reason the light is redshifted, and is also disappears because the light becomes so weak and redshifted that no practical instrument could detect its light anymore. Equation (19) says that all of these effects (the redshifting/slowing and the weakening) grow exponentially quickly according to the hovering observer's clock. For the curved spacetime of a black hole, the quantitative details are different (article 51186), but the conclusion is qualitatively the same: the dropped beacon disappears exponentially quickly according to an hovering observer hovering outside the black hole.

13 The local flatness theorem

As an example of the local flatness theorem (article 48968), this section expands the metric (7) about the event $(u,x,y,z) = (0,R,0,0)$ and then uses a coordinate transform to eliminate the linear terms. Expanding $r^2 \equiv \mathbf{x}\cdot\mathbf{x}$ about $(x,y,z) = (R,0,0)$ gives

$$r^2 = R^2 + 2R\,\delta x + \text{quadratic terms},$$

with $\delta x \equiv x - R$. This gives

$$\frac{R}{r} = 1 - \frac{\delta x}{R} + O(1/R^2) \qquad\qquad dr = dx + \frac{\delta\mathbf{x}\cdot d\mathbf{x}}{R} + O(1/R^2)$$

with $\delta\mathbf{x} \equiv (\delta x,\, y,\, z)$. Use this in (7) to get

$$\begin{aligned} d\tau^2 &= dw^2 - d\mathbf{x}^2 - \left(1 - \frac{\delta x}{R}\right)\left(dw + dx + \frac{\delta\mathbf{x}\cdot d\mathbf{x}}{R}\right)^2 + O(1/R^2) \\ &= -2du\,dx - dy^2 - dz^2 + \frac{\delta x}{R}du^2 - 2\frac{\delta\mathbf{x}\cdot d\mathbf{x}}{R}du + O(1/R^2) \end{aligned}$$

with $u \equiv w + x$. The linear terms can be eliminated by writing the line element in terms of a new coordinate system (U,X,y,z), with U,X defined implicitly by

$$u = U - \frac{U^2}{4R} \qquad\qquad \delta x = X - \frac{X^2+y^2+z^2}{2R} + \frac{XU}{2R}.$$

This leaves

$$d\tau^2 = -2dU\,dX - dy^2 - dz^2 + O(1/R^2),$$

which is the line element for flat spacetime, modulo terms of order $1/R^2$. This shows that spacetime is approximately flat in a sufficiently small (in units of R) neighborhood of any event on the horizon.[13] Within a region of spacetime of size ~ 1 km (in all four coordinates) that straddles the event horizon of a black hole with $R = 1000$ km, the deviation from perfect flatness is only $\sim 10^{-6}$.

[13] It's approximately flat in a sufficiently small neighborhood of any other event, too (if $r > 0$), but the message here is that the horizon is locally just like anywhere else, even though it is plays a special role on larger scales.

14 Gravitational redshift

The line element (7) is invariant under translations of the w coordinate. Thanks to this symmetry, we can immediately infer – without any calculation – that an object with a worldline of the form $(w, \mathbf{x}) = (\lambda, \text{constant})$ is hovering at a fixed location relative to the central body.

Now consider two such objects, with these worldlines:

$$\begin{aligned} &\text{Worldline 1: } (w, x, y, z) = (\lambda_1, x_1, 0, 0) \\ &\text{Worldline 2: } (w, x, y, z) = (\lambda_2, x_2, 0, 0) \end{aligned}$$

with constants $x_2 > x_1 > R$, so that both worldlines are timelike. Suppose that object 1 emits a pulse of light every ϵ seconds according to object 1's own proper time. At what rate are these pulses received by object 2 according to object 2's proper time?

To analyze this, we can begin by relating each object's proper time to the w-coordinate. Use these two worldlines in equation (7) to get

$$d\tau_k^2 = \left(1 - \frac{R}{x_k}\right) dw^2$$

for $k \in \{1, 2\}$. These equations say that a proper time interval of $\delta\tau_k$ on object k's clock corresponds to a w-coordinate interval of

$$(\delta w)_k = \frac{\delta\tau_k}{\sqrt{1 - R/x_k}}. \tag{20}$$

Now, let $(\delta w)_{12}$ is the difference in w-coordinates between a reception event and the corresponding emission event. Then if w_1 is the w-coordinate of an emission even, the w-coordinate of the corresponding reception event is

$$w_2 = w_1 + (\delta w)_{12}. \tag{21}$$

Thanks again to the w-translation symmetry, $(\delta w)_{12}$ is independent of w_1, so equation (21) implies that the w-coordinate interval between successive reception events is the same as the w-coordinate interval between successive emission events:

$$(\delta w)_2 = (\delta w)_1.$$

Combine this with (20) to infer

$$\delta\tau_2 = \sqrt{\frac{1 - R/x_2}{1 - R/x_1}}\,\delta\tau_1.$$

Finally, use the assumed conditions $x_2 > x_1 > R$ to deduce $\delta\tau_2 > \delta\tau_1$. In words: the rate at which object 1's clock is ticking is slower according to object 2 (which is farther away from the central body) than it is according to object 1.

This phenomenon is often called **gravitational redshift** (as I did in the title of this section), but we really should simply call it an *example of redshift*. Separating non-gravitational and gravitational redshift effects is not possible in general, at least not in any natural and systematic way. It might seem to be possible for metrics with enough symmetry, like the Schwarzschild metric (7), but concepts that might seem to make sense in highly symmetric special cases often don't make sense in generic non-symmetric cases. The popular idea that redshift can be decomposed into gravitational and non-gravitational contributions is one of those concepts.

15 Gravitational redshift: analog in flat spacetime

A similar phenomenon occurs in flat spacetime. To see this, start with the standard line element of flat spacetime:

$$d\tau^2 = dw^2 - (dx^2 + dy^2 + dz^2). \tag{22}$$

Define new coordiantes W, X by the conditions[14]

$$\begin{aligned} w &= X\sinh(aW) \\ x &= X\cosh(aW) \end{aligned}$$

for some constant $a > 0$, and substitute these into (22) to get this alternative form for the line element of flat spacetime:

$$d\tau^2 = (aX)^2 dW^2 - (dX^2 + dy^2 + dz^2). \tag{23}$$

Now consider two objects with worldlines

$$\begin{aligned} &\text{Worldline 1: } (W, X, y, z) = (\lambda_1, X_1, 0, 0) \\ &\text{Worldline 2: } (W, X, y, z) = (\lambda_2, X_2, 0, 0) \end{aligned}$$

with constants $X_2 > X_1 > 0$. Use these in (23) to get

$$d\tau_k^2 = (aX_k)^2 dW^2$$

for $k \in \{1, 2\}$. The line element (23) has W-translation symmetry, so we can use the same reasoning as in the previous section to infer

$$(\delta\tau)_2 = \frac{X_2}{X_1}(\delta\tau)_1,$$

and then the assumed condition $X_2 > X_1 > 0$ implies $\delta\tau_2 > \delta\tau_1$, as before.

[14] The new coordinate system W, X, y, z is another example of Rindler coordinates.

16 A more traditional form of the line element

Because of the factor $(dw + dr)^2$, the line element (7) has a cross-term $dw\,dr$, so g_{ab} is not diagonal. The same spacetime geometry is traditionally expressed in a different form, using coordinates in which the metric is diagonal. This section derives the more traditional form.

The traditional form can be derived from (7) by a coordinate transformation. The coordinates $\mathbf{x} = (x, y, z)$ remain the same, but the coordinate w is rewritten in terms of a new coordinate t like this:

$$w = t + f(r), \tag{24}$$

where the function $f(r)$ is chosen so that the new line element doesn't have a cross-term $dt\,dr$. To derive the function $f(r)$ that achieves this goal, start by taking the differential of the preceding equation to get

$$dw = dt + f'(r)\,dr$$

where $f'(r)$ is the derivative of $f(r)$ with respect to r. Substitute this into (7) to get

$$d\tau^2 = \big(dt + f'(r)\,dr\big)^2 - d\mathbf{x}\cdot d\mathbf{x} - \frac{R}{r}\big(dt + f'(r)\,dr + dr\big)^2,$$

and re-arrange to get

$$\begin{aligned} d\tau^2 = {} & \left(1 - \frac{R}{r}\right) dt^2 - d\mathbf{x}\cdot d\mathbf{x} + \left((f')^2 - \frac{R}{r}(f'+1)^2\right) dr^2 \\ & + 2\left(f' - \frac{R}{r}\big(f'+1\big)\right) dt\,dr. \end{aligned} \tag{25}$$

The coefficient of the cross-term $dt\,dr$ is zero if and only if we take the function $f(r)$ to satisfy

$$f'(r) = \frac{R}{r - R}, \tag{26}$$

which is solved by

$$f(r) = \begin{cases} R\log\frac{r-R}{R} & \text{if } r > R \\ -R\log\frac{R-r}{R} & \text{if } r < R \end{cases}$$

where "log" denote the natural log.[15] This shows that the desired coordinate transform (24) is[16]

$$\boxed{t = \begin{cases} w - R\log\frac{r-R}{R} & \text{if } r > R \\ w + R\log\frac{R-r}{R} & \text{if } r < R \end{cases}} \tag{27}$$

Substitute (26) into (25) to get[17]

$$\boxed{d\tau^2 = \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - (d\mathbf{x}\cdot d\mathbf{x} - dr^2).} \tag{28}$$

The independent coordinates are now t and $\mathbf{x} = (x, y, z)$, and r is still the function defined by (5). This is the traditional form of the line element, except that the "angular" part is written in terms of $\mathbf{x}$ instead of using spherical coordinates (section 3). However we write it (as (28) or as (7)), the metric defined by this line element is called the **Schwarzschild metric**.

[15]Sometimes the natural log is denoted "ln" instead, because engineers have a tradition of writing "log" for the base-10 log. I prefer to use "log" for the natural log, because it's easier to read, and because if we ever want to use something unnatural like a base-10 log, then we can indicate it explicitly with a subscript, just like we would indicate any other unnatural base.

[16]This is a special case of the family of coordinate transforms considered in Gaur and Visser (2023), which they use to generate other common and uncommon-but-interesting representations of the same spacetime geometry.

[17]Article 99922 uses general relativity (the Einstein field equation) to show that in N-dimensional spacetime, the ratio R/r in equation (28) should be generalized to $(R/r)^{N-3}$.

17 Discontinuity of the traditional coordinates

The coordinate transform (27) is really *two different* coordinate transformations, one that is valid only for $r > R$ and another that is valid only for $r < R$. Both transformations diverge as $r \to R$, so we can't use these coordinate systems to analyze things that involve events on the horizon, such as when an object falls across the horizon. That's why I started with the line element (7) instead: it's well-defined for all $r > 0$, including $r = R$.

Following tradition, I used the same letters for both of the coordinate systems defined by (27), one for $r > R$ and one for $r < R$, but this tradition can be misleading. To highlight just how misleading it can be, consider the "worldline" defined by

$$(t, x, y, z) = (0, \lambda, 0, 0) \qquad\qquad \frac{R}{2} < \lambda < 2R. \tag{29}$$

I wrote "worldline" in scare-quotes because even though it looks like it should be a single continuous worldline, it's actually two separate worldlines that don't even come close to intersecting each other! That's because the condition $t = 0$ means two very different things depending on whether $r < R$ or $r > R$. When we rewrite (29) in terms of the original w, x, y, z coordinate system (in which the Schwarzschild metric (7) is well-defined for all $r > 0$), it becomes

$$(w, x, y, z) = \begin{cases} \big(+R\log\frac{\lambda-R}{R},\ \lambda,\ 0,\ 0\big) & r > R, \\ \big(-R\log\frac{R-\lambda}{R},\ \lambda,\ 0,\ 0\big) & r < R. \end{cases} \tag{30}$$

This is *two* worldlines: the one with $r > R$ has $w \to -\infty$ at the horizon, and the one with $r < R$ has $w \to +\infty$ at the horizon. The "worldline" (29) isn't even close to being a single continuous worldline.

This illustrates just how misleading the traditional form (28) can be. It uses two different coordinate systems, one for $r > R$ and one for $r < R$, misleadingly using the same letters t, x, y, z for both of them. The traditional form is fine for studying the geometry of spacetime outside the earth (where $r \gg R$) but not for studying the geometry of spacetime across the event horizon ($r = R$).

18 White hole

Consider these two different-looking line elements:

$$d\tau^2 = dw^2 - d\mathbf{x}\cdot d\mathbf{x} - \frac{R}{r}(dw + dr)^2 \tag{31}$$

$$d\tau^2 = d\overline{w}^2 - d\mathbf{x}\cdot d\mathbf{x} - \frac{R}{r}(d\overline{w} - dr)^2. \tag{32}$$

The independent coordinates are $w, \mathbf{x}$ in equation (31) and are $\overline{w}, \mathbf{x}$ in equation (32), with $r \equiv \sqrt{\mathbf{x}\cdot\mathbf{x}}$ in both cases. Equation (31) is the same as (7). Equation (32) looks different, but it actually defines the same geometry as (31) wherever $r > R$ if we take the coordinates $w, \overline{w}$ to be related to each other by[18]

$$w = \overline{w} + 2R\log\frac{r-R}{R}. \tag{33}$$

We can view the pair of equations (31) and (32) as defining the geometry of a manifold that is partly covered by the $w, \mathbf{x}$ coordinate system and partly covered by the $\overline{w}, \mathbf{x}$ coordinate system.[19] The two coordinate systems overlap each other only where $r > R$.

The only difference in the forms of equations (31) and (32) is the sign inside the last term, so the analysis in sections 7-10 can easily be adapted to (32). The conclusion is that (32) describes a **white hole**: causal worldlines starting at $r < R$ are necessarily *outgoing*, the opposite of a black hole. Just like nothing can escape from a black hole, nothing can *enter* a white hole. Taken together, equations (31) and (32) define the geometry of a single spacetime that contains both a black hole and a white hole. The white-hole part is an artifact of considering an *eternal* black hole: it would be absent if we considered the metric of a black hole that formed from the gravitational collapse of ordinary matter.

[18]To prove this, substitute this expression for w into equation (31) and observe that the resulting equation can be reduced to (32).

[19]Actually, the $w, \mathbf{x}$ and $\overline{w}, \mathbf{x}$ coordinate systems together still don't cover the whole manifold. Referring to the picture on page 21 in Townsend (1997): the $w, \mathbf{x}$ coordinate system covers regions I and II, and the $\overline{w}, \mathbf{x}$ coordinate system covers regions I and III. Neither covers region IV.

19 Things this article didn't address

- Real black holes aren't always black. **Accretion** can make black holes very bright.
- This article considered only a non-rotating black hole. Real black holes are expected to be rapidly rotating. The metric of an an *ideal*, perfectly-symmetric and eternal rotating black hole is called the **Kerr metric**.
- This article considered only an eternal black hole – one that has been around forever. Real black holes haven't been around forever, and they can grow by consuming more matter and by merging with other black holes.[20]
- The title of this article used the adjective "ideal," partly because realistic black holes form under conditions that don't have perfect symmetry. Under realistic conditions, an effect called the **mass inflation instability** tends to significantly modify the metric behind the event horizon.[21] In particular, whereas an ideal eternal rotating black hole would have an **inner horizon** that would act like a one-way portal to another "universe," the mass inflation instability replaces the inner horizon with a singularity. A singularity, in turn, is a symptom that we have exceeded the limits of general relativity's validity: general relativity ignores quantum effects, which undoubtedly become important under such extreme conditions.
- Finally, this article ignored quantum effects, which can also have interesting consequences on a larger scale, not just near a singularity. On a larger scale, quantum effects cause a black hole to radiate like a thermal body

[20]This doesn't contradict the concept of an event horizon. In N-dimensional spacetime, an event horizon is the $(N-1)$-dimensional submanifold that separates two types of events: those that can be connected to **future null infinity** by a lightlike geodesic, and those that cannot (section 12.1 in Wald (1984)). Nothing in this definition prevents black holes from growing or merging, even though the simple metric considered in this article didn't illustrate those possibilities.

[21]For an online introduction to the mass inflation instability, I recommend Poisson (1990). That review focuses on charged black holes, but it comments on the relevance of the results to rotating black holes.

with an extremely low temperature. (The more massive the black hole, the lower the temperature.) This is called **Hawking radiation**. This would cause the black hole to eventually evaporate if it weren't consuming anything, but the black hole would still last $\sim G^2M^3/(c^4\hbar)$, ignoring a numeric factor that doesn't change the message here.[22] For a solar-mass black hole, $G^2M^3/(c^4\hbar) \sim 10^{63}$ years.

[22] G is the gravitational constant, M is the mass, c is the speed of light, and $\hbar$ is Planck's constant.

20 References

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