Constructing Principal Bundles from Patches

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Abstract The concept of a principal G-bundle over a base space M is the mathematical foundation for the concept of a gauge field, where G is the gauged group and M is space or spacetime. A trivial principal bundle $M \times G \to M$ exists for every combination of G and M. Nontrivial principle bundles exist for some combinations of G and M but not for others. When they do exist, they may be constructed using what this article calls **patches** – trivial principal G-bundles over parts of the base space M, glued together using **transition functions**, also called **clutching functions**. This article uses that approach to derive some results about the (non)existence of nontrivial principal G-bundles when G is a compact Lie group and when the base space is an n-dimensional sphere or an n-dimensional torus, for various values of n.

Contents

1 Introduction

This article explores the existence of principal G-bundles when the base space is an n-dimensional sphere or an n-dimensional torus, where G is a compact Lie group.[1,2]

If two principal G-bundles over a given base space are isomorphic to each other, then this article won't distinguish between them. The definition of **isomorphic** won't be reviewed here,[3] but it roughly means that they are the same when regarded as abstract fiber bundles, even if they are implemented differently. A little less roughly, it means that their total spaces are homeomorphic to each other and that their projections and G-actions are consistent with that homeomorphism.

For given G and B, principal G-bundles over a base space B are said to be **classified by** a set ω if elements of ω correspond one-to-one with isomorphism classes of principal G-bundles over B. If B is a CW complex,[4] then principal G-bundles over B are classified by the homotopy set[5] $[B, BG]$, where BG is a topological space called a **classifying space** for G.[3] The homotopy groups of BG are determined by those of G:[3]

$$\pi_n(BG) \simeq \pi_{n-1}(G). \tag{1}$$

Instead of classifying principal G-bundles over B, the goal in this article is more modest: the goal is only to determine whether nontrivial principal G-bundles over B exist. Section 2 will summarize the results.

[1] Article 70621 introduces the concept of a **principal G-bundle**.

[2] Section 19 is an exception: it compares the groups $G = U(1)$ and $G = \mathbb{R}$. The group $\mathbb{R}$ is noncompact. This comparison is relevant to quantum electrodynamics (article 51376).

[3] Article 35490

[4] Article 93875 defines **CW complex**

[5] Section 5 will review the concept of a *homotopy set*.

2 Summary of results

To determine when nontrivial principal G-bundles over a base space B exist, this article considers how bundles over B may be constructed from trivial bundles over parts of B that collectively cover B. This will be done for various combinations of G and B, where B is either a sphere (S^n) or a torus (T^n). This table summarizes the results, using "yes" to indicate that nontrivial principal G-bundles over B exist, and using "-" to indicate that all principal G-bundles over B are trivial.

	B (the base space)							
	S^1	S^2	S^3	S^4	S^1	T^2	T^3	T^n, $n \geq 4$
G discrete	yes	-	-	-	yes	yes	yes	yes
$G = U(1)$	-	yes	-	-	-	yes	yes	yes
$G = SO(k)$, $k \geq 3$	-	yes	-	yes	-	yes	yes	yes
$G = SU(k)/\mathbb{Z}_k$, $k \geq 2$	-	yes	-	yes	-	yes	yes	yes
$G = SU(k)$, $k \geq 2$	-	-	-	yes	-	-	-	yes

The case S^1 is listed twice because it's both a sphere and a torus.

When the base space is a sphere, published information about principal bundles is relatively abundant.[6] When the base space is a torus, published information about principal bundles is not as easy to find (or at least is not as accessible to physicists), so this article devotes more effort to them.

[6] Section 8 will show that principal G-bundles over spheres are classified by homotopy groups $\pi_j(G)$, and abundant information is available about the homotopy groups $\pi_j(G)$ of a Lie group G (article 92035).

3 Conventions and notation

- *Map* means continuous map, and *function* means continuous function.
- Each topological space is assumed to be homeomorphic to a CW complex.[7,8]
- S^n is an n-dimensional sphere, also called an ***n*-sphere**.
- T^n is an n-dimensional torus (the cartesian product of n circles), also called an ***n*-torus**.
- $\mathbb{Z}$ is the integers, $\mathbb{Z}_k$ is the integers modulo k, and $\mathbb{R}$ is the real numbers.
- If X and Y are topological spaces, then $[X,Y]$ is the set of homotopy classes of maps from X to Y. The set of basepoint-preserving homotopy classes of maps is denoted $[X,Y]_0$. Article 69958 reviews the definitions.
- A group or homotopy set is called **trivial** if it has only one element.
- $\pi_j(X)$ is the jth homotopy group of a topological space X, and $H^j(X;\mathbb{Z})$ is the jth integer cohomology group.[9]
- A topological space X is called ***n*-connected** if $\pi_j(X)$ is trivial for all $j\leq n$. In particular, **1-connected** means $\pi_0(X)$ and $\pi_1(X)$ are both trivial. The word **connected** by itself is an abbreviation for 0-connected.
- $U(k)$ and $SU(k)$ are the unitary and special unitary groups.
- $O(k)$ and $SO(k)$ are the orthogonal and special orthogonal groups.
- If G and H are groups, then $G\simeq H$ means G and H are isomorphic to each other. If X and Y are topological spaces, then $X\simeq Y$ means X and Y are homeomorphic to each other.
- BG is a **classifying space**[10] for the group G.

[7] Article 93875

[8] Every smooth manifold is homeomorphic to a CW complex (article 93875).

[9] Articles 61813 and 28539

[10] Article 35490

4 A general result for dimensions 1,2,3

An n-dimensional **CW complex** B is a union of k-cells with dimensions $k \in \{0, 1, 2, ..., n\}$, involving any number of k-cells of each dimension that are joined together according to some natural rules.[11] Every n-dimensional topological manifold with $n \in \{1, 2, 3\}$ is homeomorphic to a CW complex.[11,12] This section shows that if G is a 1-connected compact Lie group and the base space B is an n-dimensional CW complex with $n \in \{1, 2, 3\}$, then all principal G-bundles over B are trivial.[13] In particular, all principal $SU(k)$-bundles over S^1, S^2, S^3, T^2, and T^3 are trivial.[14]

The strategy will be to show that if G is a 1-connected Lie group, then any principal G-bundle over a CW complex of dimension ≤ 3 admits a section.[15] The result then follows from the fact that a principal bundle is trivial if and only if it admits a section.[16]

The ***k*-skeleton** of a CW complex is the union of the j-cells with $j \leq k$. Choose any $k < n$. Suppose that the kth homotopy group $\pi_k(G)$ is trivial, and suppose that any principal G-bundle over the k-skeleton admits a section. Under those conditions, any principal G-bundle over the $(k+1)$-skeleton also admits a section. To deduce this, start with a section over the k-skeleton. The boundary of any $(k+1)$-cell is homeomorphic to S^k, so premise that $\pi_k(G)$ is trivial means that any section over the boundary of a $(k+1)$-cell may be extended to a section over the whole $(k+1)$-cell. Doing this for all of the $(k+1)$-cells gives a section over the whole $(k+1)$-skeleton.

Now, suppose that B is an n-dimensional CW complex with $n \in \{1, 2, 3\}$.

[11] Article 93875

[12] Example: the 3-torus $T^3 \equiv S^1 \times S^1 \times S^1$ may be represented as a cube with opposite faces identified with each other. The interior of the cube is a 3-cell, each pair of opposite faces is a 2-cell, each quadruple of parallel edges (which are all identified with each other) is a 1-cell, and the 8 corners (which are all identified with each other) are a 0-cell.

[13] Dijkgraaf and Witten (1990), section 1, page 394 (for $n = 3$); Witten (1992), beginning of section 4.1 (for $n = 2$); Kubyshin (1999), equation (26) and more explicitly in the paragraphs after equation (31) (for $n = 2$); Kirk (1993), beginning of section 2.1 (for $n = 2, 3$)

[14] $SU(k)$ is 1-connected for all $k \geq 1$ (article 92035).

[15] This is one of the methods described in `https://math.stackexchange.com/questions/2856191/`.

[16] Article 70621

The premise that G is 1-connected means that $\pi_0(G)$ and $\pi_1(G)$ each have only one element, and the additional premise that G is a Lie group implies that $\pi_2(G)$ also has only one element.[17] Any principal G-bundle over the union of the 0-cells automatically admits a section. According to the previous paragraph, the fact that $\pi_0(G)$ is trivial implies that any given section over the union of the 0-cells can be extended to the whole 1-skeleton, the fact that $\pi_1(G)$ is trivial implies that it can be further extended to the whole 2-skeleton (relevant if $n \in \{2, 3\}$), the fact that $\pi_2(G)$ is trivial implies that it can be further extended to the whole 3-skeleton (relevant if $n = 3$). The result is a section over all of B, and the fact that a section exists shows that the principal G-bundle over B is trivial, as claimed.

When $n = 1$, this approach still works even if G is merely connected (so that $\pi_0(G)$ is trivial). Different approaches are needed when $n \geq 4$, or when $n \geq 2$ is G is not 1-connected, or when $n \geq 1$ if G is not even connected.

[17] Article 92035

5 Homotopy

Intuitively, a **homotopy** from one map $f : X \to Y$ to another map $g : X \to Y$ is a continuous deformation from $f(X)$ to $g(X)$ within Y. If a homotopy exists between f and g, then f and g are said to be **homotopic** to each other.[18] For any given $f : X \to Y$, the set $[f]$ of all maps homotopic to f is called a **homotopy class**. A map $f : X \to Y$ is called **nullhomotopic** if it's homotopic to a **constant map**, which is a map that sends all of X to a single point of Y.

When X and Y are topological spaces, $[X, Y]$ denotes the set of homotopy classes of maps from X to Y.[19] Each element of $[X, Y]$ is a homotopy class $[f]$ of maps $f : X \to Y$.

If we choose a point $x_0 \in X$ and a point $y_0 \in Y$, then a homotopy that preserves the relationship $x_0 \to y_0$ throughout the deformation process is called a **based homotopy**. The points x_0, y_0 are called **basepoints**. Each element of the **based** (or **pointed**) homotopy set $[X, Y]_0$ is an equivalence class $[f]_0$ of maps using based homotopy as the equivalence relation.[19] Homotopy groups, introduced in article 61813, may be defined as $\pi_j(X) \equiv [S^j, X]_0$.

The set $[X, Y]$ is sometimes called a **free homotopy set** to distinguish it from the based homotopy set $[X, Y]_0$. The free homotopy set $[X, Y]$ and the based homotopy set $[X, Y]_0$ are not always equal to each other, but they are in these cases, among others:[19]

- They are equal to each other when Y is 1-connected.
- They are equal to each other when Y is a connected Lie group.

Beware that the based homotopy set is often denoted $[X, Y]$ in sources that only consider based homotopies.

[18] Article 61813
[19] Article 69958

6 Building a principal bundle from trivial patches

Any fiber bundle may be constructed from patches – trivial bundles over parts of the base space, glued together using transition functions. Let $U_1, U_2, U_3, ...$ be a covering of the base space B by open sets (called **charts**), and let $E_k \equiv U_k \times F$ be the total space of a trivial bundle $E_k \to U_k$ over that part of the base space. This trivial bundle is the kth **patch**.[20] If U_j and U_k overlap, then the way the jth and kth patch's fibers are glued together is specified by a **transition function**

$$\tau_{j\to k} : U_j \cap U_k \to G,$$

where elements of the group G act as homeomorphisms from the fiber F to itself. Article 70621 describes how the functions $\tau_{j\to k}$ are used to define a fiber bundle over B and the conditions they must satisfy, including the additional condition that must be satisfied to produce a principal G-bundle.

The charts in this construction don't need to be contractible,[21] and the intersections $U_j \cap U_k$ don't need to be contractible, either. Any fiber bundle can be constructed using contractible charts with contractible intersections, but the construction doesn't rely on those conditions.

[20] This name is not standard.

[21] Husemoller (1966), chapter 5, section 2

7 Diagnosing isomorphism

If two fiber bundles are both constructed using the same set of trivial patches over the same charts, then the only way they can differ from each other is through differences in their transition functions. If their transition functions are $\tau_{j\to k}$ and $\tau'_{j\to k}$, respectively then the resulting fiber bundles are isomorphic to each other if and only if maps $\phi_j : U_j \to G$ exist for which[22,23]

$$\tau_{j\to k}(u) = \phi_j^{-1}(u)\tau'_{j\to k}(u)\phi_k(u) \tag{2}$$

for all j, k. Equation (2) only uses the restrictions of the maps ϕ_j and ϕ_k to $U_j \cap U_k$, but they must be defined throughout U_j and U_k, respectively. The fact that the map ϕ_k must be defined throughout U_k for each k is an essential part of the condition for isomorphism.

Mutually homotopic transition functions automatically satisfy this condition. To deduce this, suppose that $\tau_{j\to k}$ and $\tau'_{j\to k}$ are homotopic to each other. This implies that the function

$$\big(\tau'_{j\to k}(u)\big)^{-1}\tau_{j\to k}(u) \tag{3}$$

is nullhomotopic. Then we can satisfy equation (2) by setting $\phi_j(u) = 1$ for all $u \in U_j$ and setting $\phi_k(u)$ equal to (3) for all $u \in U_j \cap U_k$, and the fact that this part of ϕ_k is nullhomotopic implies that it can be extended to all of U_k.

This condition for isomorphism implies that a fiber bundle defined by the transition functions $\tau_{j\to k}$ is trivial if and only if maps $\phi_j : U_j \to G$ exist for which

$$\tau_{j\to k}(u) = \phi_j^{-1}(u)\phi_k(u), \tag{4}$$

because a fiber bundle whose transition function $\tau'_{j\to k}$ are all constant and equal to 1 is clearly trivial.

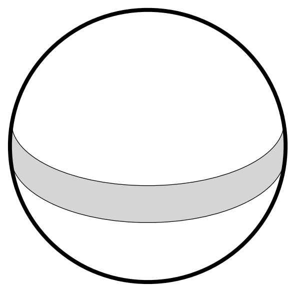
[22] Husemoller (1966), chapter 5, definition 2.6 and theorem 2.7; Steenrod (1951), section 2.10

[23] This condition is sometimes expressed in words by saying that the transition functions $\tau_{j\to k}$ and $\tau'_{j\to k}$ are **cohomologous** to each other (Calegari (2019), text below definition 1.3)

8 Principal bundles over spheres

This section shows that principal G-bundles over the n-sphere S^n are classified by the homotopy group $\pi_{n-1}(G)$.[24,25]

The sphere S^n can be covered by two open sets U_1 and U_2 that overlap each other only in a narrow neighborhood of the equator. This is illustrated here for the case $n = 2$, with the overlap $U_1 \cap U_2$ shaded gray:



The open sets U_1 and U_2 will be called **hemispheres** even though they overlap. Each hemisphere is contractible, and any fiber bundle over a contractible base space is trivial, so any nontriviality in a bundle over S^n can only come from the transition function $\tau_{1\to 2} : U_1 \cap U_2 \to G$. The overlap $U_1 \cap U_2$ can be continuously retracted onto the equator, which is topologically S^{n-1}. Mutually homotopic transition functions define isomorphic bundles, so all of the important information in the transition function is already present in its restriction to the equator S^{n-1}.

The isomorphism condition (2) requires the function $\phi_k : U_k \to G$ to be defined everywhere in U_k. Since U_k is contractible, this implies that the restriction of ϕ_k to the equator is nullhomotopic.[26] When G is a connected Lie group, this implies that the restriction of ϕ_k to the equator is homotopic to the function that maps the whole equator to the identity element of G. Then equation (2) is satisfied if and only if the transition functions $\tau_{1\to 2}$ and $\tau'_{1\to 2}$ are homotopic to each other, so

[24] Cohen (2023), theorem 4.7

[25] Section 1 defined *classified by*.

[26] Each hemisphere U_k is an $(n-1)$-dimensional ball, so we can think of a function $U_k \to G$ as a homotopy $S^{n-1} \times [0, 1] \to G$ that maps $S^{n-1} \times \{1\}$ to a single point in G.

principal G-bundles over S^n are classified by the free homotopy set $[S^{n-1}, G]$ when G is connected. For a connected Lie group, the free homotopy set $[S^{n-1}, G]$ is the same as the based homotopy set $[S^{n-1}, G]_0$,[27] so we can use the isomorphism[27]

$$\pi_{n-1}(G) \equiv [S^{n-1}, G]_0 \tag{5}$$

to get the result stated at the beginning of this section when G is connected.

Now suppose that the group G is not connected, and let G_0 denote the connected component that contains the identity element. The images of the functions $\phi_k : U_k \to G$ in equation (2) might not be in G_0, so equation (2) no longer requires $\tau_{1\to 2}$ and $\tau'_{1\to 2}$ to be homotopic to each other. In other words, principal G-bundles over S^n are no longer classified by the free homotopy set $[S^{n-1}, G]$. However, they are still classified by the based homotopy set $[S^{n-1}, G]_0$. To deduce this, suppose that the image of $\tau'_{1\to 2}$ does not intersect G_0, and let g be one of the points in its image. If we choose ϕ_1 and ϕ_2 to be constant functions whose images are g and the identity element, respectively, then the image of $\tau_{1\to 2}$ has at least one point in G_0.[28] This shows that if we choose a basepoint in the equator S^{n-1} and consider only transition functions $\tau_{1\to 2}$ that map this point to the identity element of G, then we can still get all possible principal G-bundles over S^n. This is why principal G-bundles over S^n are classified by (5) even when G is not connected.

[27] Section 5

[28] The 0-sphere S^0 consists of two points that are not connected to each other, so when $n = 1$, the image of $\tau_{1\to 2}$ might consist of one point in G_0 and one point not in G_0.

9 Principal bundles over spheres: examples

Section 8 showed that principal G-bundles over the n-sphere S^n are classified by the homotopy group $\pi_{n-1}(G)$. Examples:

- If G is a discrete group, then $\pi_{n-1}(G)$ has only one element for each $n \geq 2$, so all principal G-bundles over S^n are trivial in these cases.
- If G is a discrete group with k elements, then $\pi_0(G)$ also has k elements, so $k-1$ nontrivial principal G-bundles over the circle S^1 exist.
- If G is a connected group, then $\pi_0(G)$ only has one element, so all principal G-bundles over S^1 are trivial in this case.[29]
- $\pi_1(U(1)) = \mathbb{Z}$, so nontrivial principal $U(1)$-bundles over S^2 exist.[30]
- $\pi_{n-1}(U(1))$ has only one element for each $n \geq 3$,[31] so all principal $U(1)$-bundles over S^n are trivial when $n \geq 3$.
- If $k \geq 3$, then $\pi_1(SO(k))$ has two elements,[31] so (up to isomorphism) one nontrivial principal $SO(k)$-bundle over S^2 exists for each $k \geq 3$.[32]
- If G is a 1-connected Lie group, then $\pi_1(G)$ and $\pi_2(G)$ each have only one element,[33] so nontrivial G-bundles over S^2 or S^3 do not exist. This is consistent with the general result in section 4.
- If G is a compact simple[34] Lie group, like $SU(k)$ is, then $\pi_3(G) = \mathbb{Z}$,[31] so infinitely many non-isomorphic nontrivial principal G-bundles over S^4 exist.

[29] The Klein bottle is a nontrivial circle bundle over a circle, but it's not a principal $U(1)$-bundle: multiplying a circular fiber by an element of $U(1)$ is the same as rotating the circle around its axis, and it can't rotate the fibers of a Klein bottle in a consistent direction at all points of the base space.

[30] This includes the **Hopf fibration**. Article 03838 studies these bundles in more detail.

[31] Article 92035

[32] Maxim (2018), section 4, exercise 2 (for $k = 3$)

[33] For $\pi_1(G)$, this is part of the definition of *1-connected*. For $\pi_2(G)$, this is a special property of Lie groups (article 92035).

[34] A Lie group is called **simple** if it doesn't have any connected normal subgroups (article 92035).

10 Example: principal bundles over S^1

This section illustrates the reasoning in section 8 using one of the easiest examples: if G is connected, then all principal G-bundles over a circle S^1 are trivial.

The "equator" S^0 of S^1 is a pair of points. Let p and p' be these two points. The base space S^1 may be covered by a pair of line segments U_1 and U_2 that overlap each other only at their endpoints – only on the equator S^0:

$$U_1 \cap U_2 = \{p, p'\}.$$

A transition function $\tau'_{1\to 2} : S^0 \to G$ chooses two elements of G, one for each of the two points $p, p' \in S^0$. Call these two elements g and g'. The functions ϕ_1 and ϕ_2 in (2) each map one of the line segments into G. If G is connected, then we can choose these maps to satisfy[35]

$$\phi_1(U_1) = g^{-1} \qquad \phi_2(p) = 1 \qquad \phi_2(p') = (g')^{-1}g,$$

and then the new transition function $\tau_{1\to 2}$ maps both of the points p, p' to the identity element of G, so the resulting bundle is trivial.

The relationship (1) leads to the same conclusion. If the group G is connected ($\pi_0(G) = 0$), then BG is 1-connected ($\pi_1(BG) = \pi_0(BG) = 0$). Every map from S^1 into a 1-connected space is homotopic to a trivial map (one that maps S^1 to a single point), so $[S^1, BG] = 0$. Combine this with the correspondence between $[X, BG]$ and principal G-bundles over X to finish the proof.

A nontrivial bundle with connected fiber G over S^1 may still exist, but it can't be a *principal* G-bundle. Example: the Klein bottle is a nontrivial bundle over S^1 with fiber $S^1 \simeq U(1)$, but it's not a principal $U(1)$-bundle because the transition function cannot be implemented using only multiplication by elements of the group $U(1)$.

[35]The condition that G is connected is needed to ensure the existence of a continuous map $\phi_2 : U_2 \to G$ that takes the values at the endpoints $p, p' \in U_2$, with no restrictions on g or g'.

11 Extending a principal bundle from T^n to T^{n+1}

This section shows that if a nontrivial principal G-bundle over an n-dimensional torus T^n exists for a given G and n, then a nontrivial principal G-bundle over T^{n+1} also exists.[36]

Suppose that a nontrivial principal G-bundle over a given CW complex M exists. This implies the existence of a map $f : M \to BG$ that is not homotopic to a constant map. Now let M' be another CW complex, and define $f' : M \times M' \to BG$ by

$$f'(m, m') \equiv f(m)$$

for all $m \in M$ and $m' \in M'$. Let I denote the interval $[0, 1] \subset \mathbb{R}$. If f' were homotopic homotopic to a constant map, then (by definition of homotopic)[37] a function $h' : M \times M' \times I \to BG$ would exist with

$$h'(m, m', 0) = f'(m, m') = f(m) \qquad\qquad h'(m, m', 1) = \text{point}.$$

Then for any given point $m'_0 \in M'$, the function $h : M \times I \to BG$ defined by $h(m, t) \equiv h'(m, m'_0, t)$ would be a homotopy from f to a constant map, which would contradict our premise about f. This shows that f' cannot be homotopic to a constant map, which implies that a nontrivial principal G-bundle over $M \times M'$ exists.[38]

Set $M = T^n$ and $M' = S^1$ to get the result stated at the beginning of this section.

[36] This way of extending a bundle's base space from M to $M \times M'$ might not account for all isomorphism classes of bundles over $M \times M'$, but classifying all of them is not the goal here.

[37] Article 61813

[38] Article 35490, reviewed in section 1

12 Principal bundles over a torus from trivial patches

This section shows that if all principal G-bundles over a connected space M are trivial, then a nontrivial principal G-bundles over $M \times S^1$ exists if and only if the free homotopy set $[M, G]$ has more than one element.

Choose a Lie group G, and suppose we have already determined that all principal G-bundles over M are trivial.[39] If we represent the S^1 factor in $M \times S^1$ as $\mathbb{R}$ modulo 2π, then the new base space $M \times S^1$ may be covered with two open sets $U_1 = M \times I_1$ and $U_2 = M \times I_2$, where I_1 and I_2 are the intervals

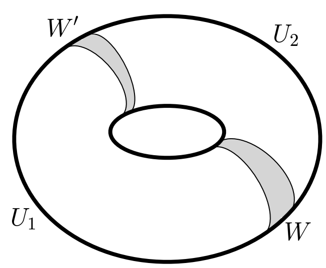
$$I_1 = [-\epsilon,\, \pi + \epsilon] \qquad\qquad I_2 = [\pi - \epsilon,\, 2\pi + \epsilon]$$

for some $0 < \epsilon \ll 1$ so that $I_1 \cup I_2 = S^1$. If every principal G-bundle over M is trivial, then every principal G-bundle over $M \times I_k$ is also trivial,[40] because $M \times I_k$ can be continuously retracted to $M \times (\text{point}) \simeq M$. The goal is to determine when we can build a nontrivial bundle over $M \times S^1$ from these two trivial patches.[41]

The overlap $U_1 \cap U_2$ has two components that are not connected to each other:

$$W \equiv M \times [-\epsilon, \epsilon] \qquad\qquad W' \equiv M \times [\pi - \epsilon, \pi + \epsilon].$$

This is illustrated below for the case $M = S^1$, so that $M \times S^1$ is a 2d torus:



Each of the open sets U_k covers a little more than half of the torus, and the two shaded regions are the two regions W and W' where U_1 and U_2 overlap. The bundles

[39] This condition on M is in effect throughout this section.

[40] Homotopy-equivalent base spaces admit the same isomorphism classes of principal G-bundles (article 35490).

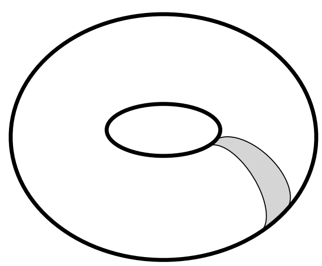
[41] The base spaces $M \times I_k$ for these patches are not contractible, and that's okay (section 6).

over U_1 and U_2 are trivial, so the topology of the bundle over $U_1 \cup U_1 = M \times S^1$ is defined by a transition function

$$\tau_{1\to 2} : U_1 \cap U_2 \to G. \tag{6}$$

We can think of the transition function (6) as a pair of transition functions, one with domain W and one with domain W'. Without loss of generality, we may assume that the transition function in one of these regions, say W', is the trivial function that maps all of W' to the identity element of G. To understand why this doesn't lose any generality, suppose that the other ends (the ends that overlap in the other region W) were left un-glued. Then the resulting bundle would have a base space of the form $M \times I$, where I is an interval, and this bundle must still be trivial for the same reason that each of the two patches is trivial.

After choosing the transition function in W' to send all of W' to the identity element of G, the bundle over $M \times S^1$ is determined by the transition function in the other overlap-region W, illustrated here for the case $M = S^1$ again:



We chose the transition function so that the functions ϕ_1 and ϕ_2 in equation (4) are equal to each other within W', which implies that that their restrictions to W must be homotopic to each other.[42] Given a transition function $\tau_{1\to 2}$ in W, the resulting principal G-bundle over $M \times S^1$ is trivial if and only if $\tau_{1\to 2}$ may be written as in equation (4). In the present case, saying that $\tau_{1\to 2}$ may be written as in equation (4) is equivalent to saying that $\tau_{1\to 2}$ is homotopic to the map that sends all of M to the identity element of G, because ϕ_1 and ϕ_2 are homotopic to each

[42] This is essentially the definition of homotopy, because their restrictions to W are homotopic to their restrictions to the ends of $M \times I$ (after using $\phi_1 = \phi_2$ in W').

other. Nontrivial principal G-bundles over $M \times S^1$ exist if and only if maps $M \to G$ exist that are not homotopic to that one, which is equivalent to saying that the homotopy set $[W, G]$ has more than one element. The homotopy sets $[W, G]$ and $[M, G]$ are equal because M is a deformation retract of W, so nontrivial principal G-bundles over $M \times S^1$ exist if and only if $[M, G]$ has more than one element, as claimed at the beginning of this section.[43]

We can't say that principal G-bundles over $M \times S^1$ are classified by $[M, G]$, though. Suppose that G is a not-necessarily-abelian group with a finite number N of elements. Then $[M, G]$ also has N elements, because we're assuming that M is connected. In that case, the fact that the restrictions of ϕ_1 and ϕ_2 to W must be homotopic to each other implies that they must also be equal to each other. Then equation (2) says that two transition functions define isomorphic principal bundles if and only if they are related to each other by

$$\tau_{1\to 2} = g^{-1}\tau'_{1\to 2}g \tag{7}$$

for some $g \in G$. If G is nonabelian, then (7) does not imply $\tau_{1\to 2} = \tau'_{1\to 2}$, so the number of isomorphism classes of principal bundles can be less than N.

[43] In contrast to section 8, the homotopy set here is a *free* homotopy set (no basepoints). That's because here, the restrictions of ϕ_1 and ϕ_2 to W must be homotopic to each other, so the images of $\tau_{1\to 2}$ and $\tau'_{1\to 2}$ in equation (4) can't be in different connected components of G, like they could in section 8.

13 Principal bundles over T^2: first approach

Suppose that G is a connected Lie group. In this case, all principal G-bundles over S^1 are trivial,[44] so the result derived in section 12 says that nontrivial principal G-bundles over $T^2 = S^1 \times S^1$ exist if and only if $[S^1, G]$ is nontrivial. For a connected Lie group G, elements of $[S^n, G]$ correspond one-to-one with elements of $\pi_n(G)$,[45] so nontrivial principal G-bundles over T^2 exist if and only if $\pi_1(G)$ is nontrivial. Examples:

- $\pi_1(U(1)) \simeq \mathbb{Z}$, so nontrivial principal $U(1)$-bundles over T^2 exist. This implies that nontrivial principal $U(1)$-bundles over T^n exist for all $n \geq 2$.[46]

- If $k \geq 3$, then $\pi_1(SO(k))$ has two elements,[47] so nontrivial principal $SO(k)$-bundles over T^2 exist. This implies that nontrivial principal $SO(k)$-bundles over T^n exist for all $n \geq 2$.[46,48]

- If $G = SU(k)/\mathbb{Z}_k$ with $k \geq 2$,[49] then $\pi_1(G)$ has more than one element, so nontrivial principal G-bundles over T^2 exist. This implies that nontrivial principal G-bundles over T^n exist for all $n \geq 2$.[46,50]

- If G is 1-connected, then all principal G-bundles over T^2 are trivial. This agrees with the general result in section 4. In particular, all principal $SU(k)$-bundles over T^2 are trivial.

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[44] Section 9

[45] Section 5

[46] Section 11

[47] Article 92035

[48] Equation 2.6 in Nash (1983) classifies $SO(3)$ bundles over T^4.

[49] Article 92035

[50] Examples with $n = 4$ are given in Nash (1983), whose method is partially reviewed in Ray and Sen (2022).

14 Principal bundles over a punctured T^2

This section shows that if G is connected, then all principal G-bundles over a two-dimensional **punctured torus** (T^2 with one point removed) are trivial. This result will be used in section 15.

Start with a two-dimensional torus T^2. Represent the torus T^2 as a rectangular surface whose opposite sides are identified with each other. Because of these identifications, the rectangle's boundary represents a pair of circles that intersect each other at a single point, denoted $S^1 \vee S^1$.[51] This is not a manifold, but it is a 1-dimensional CW complex, so all principal G-bundles over $S^1 \vee S^1$ are trivial if G is connected.[52] The same conclusion may also be reached using equation (1). Setting $n = 1$ in that equation gives $\pi_1(BG) \simeq \pi_0(G)$, so if G is connected, then BG is 1-connected, which implies[53] that $[S^1 \vee S^1, BG]$ is trivial.

Now let p be a point in the rectangle's interior. Then the boundary $S^1 \vee S^1$ is a deformation retract of $T^2 \setminus p$, which implies

$$[T^2 \setminus p, X] = [S^1 \vee S^1, X] \tag{8}$$

for any space X. Take X to be a classifying space BG for G to infer that $T^2 \setminus p$ admits the same isomorphism classes of principal G-bundles that $S^1 \vee S^1$ does, so all principal G-bundles over the punctured torus $T^2 \setminus p$ are trivial if G is connected.

[51] Article 69958

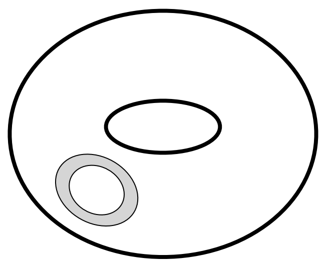
[52] Section 4

[53] To deduce this, use these relationships from article 69958: the identity $[S^1 \vee S^1, M]_0 \simeq [S^1, M]_0 \times [S^1, M]_0 = \pi_1(M) \times \pi_1(M)$, and the fact that if $[X, Y]_0$ is trivial, then $[X, Y]$ is also trivial.

15 Principal bundles over T^2: second approach

Section 13 showed that if G is connected, then nontrivial principal G-bundles over T^2 exist if and only if $[S^1, G]$ is nontrivial. This section derives the same result using a different way of assembling the torus T^2 from patches.

Think of the torus T^2 as a punctured torus whose puncture is repaired by covering it with a small disk. The shading in this picture indicates where the outer rim of the disk overlaps the inner rim of the punctured torus:



The overlap is retractible to a circle S^1, so we can again use a transition function $S^1 \to G$ to construct any principal G-bundle over T^2 from principal G-bundles over these two overlapping surfaces. A principal G-bundle over a disk must be trivial because a disk is contractible, and section 14 showed that if G is connected, then a principal G-bundle over a punctured torus must also be trivial. With those inputs, the result in section 12 shows the resulting principal G-bundle over T^2 is determined by the transition function $S^1 \to G$, just like it was in section 13.

This approach also works for any closed oriented surface M of genus $g \geq 2$.[54] For such a surface, $M \setminus$ (point) is retractible to a set of circles that all intersect each other at a single point.[55] The reasoning in section 14 again shows that every principal G-bundle over $M \setminus$ (point) is trivial if G is connected, so every principal G-bundles over M is again determined by a transition function $S^1 \to G$.[56]

[54] Ramanathan (1975), proposition 5.1

[55] This can be deduced from the fact that every closed oriented surface M of genus g is a disk with appropriate boundary-identifications. Section 1.2 in Montesinos (1987) illustrates this in the case $g = 2$.

[56] Theorem 2.2 in Oliveira (2008) gives a more general result that includes some not-necessarily-connected groups G, including $G = O(n)$, and that allows the base space to be an arbitrary 2-dimensional CW complex.

16 Principal G-bundles over T^2: third approach

Sections 13 and 15 both showed that if G is connected, then nontrivial principal G-bundles over T^2 exist if and only if $[S^1, G]$ is nontrivial. This section uses another approach to derive the same result.

Article 69958 shows that when X is 1-connected, $[T^2, X]$ and $\pi_2(X)$ are equal to each other (as sets):

$$[T^2, X] \simeq \pi_2(X). \tag{9}$$

This may be used to rederive the results stated in section 13. To do this, take X to be a classifying space BG for G. Up to isomorphism, principal G-bundles over T^2 correspond one-to-one with elements of $[T^2, BG]$.[57] The classifying space BG is not a finite-dimensional manifold, but we can always choose it to be a CW complex,[57] as assumed in the derivation of equation (9). Equation (1) gives

$$\pi_1(BG) \simeq \pi_0(G) \qquad\qquad \pi_2(BG) \simeq \pi_1(G). \tag{10}$$

The first of these equations says that equation (9) applies whenever G is connected. In that case, we can use the second of equations (10) in (9) to get

$$[T^2, BG] = \pi_2(BG) = \pi_1(G). \tag{11}$$

This shows once again that if G is connected, then nontrivial principal G-bundles over T^2 exist if and only if $[S^1, G] \simeq \pi_1(G)$ is nontrivial.

[57] Article 35490

17 Principal G-bundles over T^3

Now suppose that G is a connected and 1-connected Lie group, which implies that $\pi_0(G)$, $\pi_1(G)$, and $\pi_2(G)$ are all trivial.[58] Section 4 already showed that in this case, all principal G-bundles over T^3 are trivial. In particular, all principal $SU(k)$-bundles over T^3 are trivial.

The same conclusion may also be reached by using (9) to get

$$[T^2, G] = 0 \tag{12}$$

when G is a 1-connected Lie group. This implies[59] that we can't make a nontrivial principal G-bundle over T^3 from a trivial principal G-bundle over T^2. We already know that all principal G-bundles over T^2 are trivial when G is a 1-connected Lie group,[60] so this shows that all principal G-bundles over T^3 are trivial, too.

[58] Footnote 33 in section 9

[59] Section 12

[60] Section 13

18 Principal $SU(k)$-bundles over T^4

Suppose that G is 1-connected. Section 17 showed that in this case, all principal G-bundles over T^3 are trivial. According to section 12, if all principal G-bundles over T^3 are trivial, then nontrivial principal G-bundles over T^4 exist if and only if the free homotopy set $[T^3, G]$ is nontrivial.

The easiest example of a 1-connected compact lie group is $SU(2)$. The fact that $[T^3, SU(2)]$ is nontrivial follows from the homeomorphism

$$SU(2) \simeq S^3$$

together with the fact that $[M, S^n] \simeq \mathbb{Z}$ for any closed n-dimensional manifold M.[61] This shows that nontrivial principal $SU(2)$-bundles over T^4 exist.[62]

Article 69958 shows that $[T^3, SU(k)]$ is nontrivial for each $k \geq 2$. Altogether, this shows that nontrivial principal $SU(k)$-bundles over T^4 exist for each $k \geq 2$. The result in section 11 extends this to all bases spaces T^n with $n \geq 4$.

[61] Article 69958

[62] https://mathoverflow.net/q/195592/ and https://math.stackexchange.com/q/4420621/ describe another approach for principal $SU(2)$-bundles over four-dimensional manifolds.

19 The Lie algebra is not enough

Much of the information about a Lie group G is encoded in its Lie algebra, but not all of it. The existence of nontrivial principal G-bundles over a given base space M depends on the topology of the Lie group G, not just on its Lie algebra. This section highlights examples of that phenomenon.

- Nontrivial principal G-bundles exist over T^n if $G = \mathbb{Z}_2$ but obviously not if G is the trivial group with only one element, even though both groups have the same (trivial) Lie algebra.

- Nontrivial principal G-bundles over the sphere S^2 exist for $G = SO(3)$ but not for $G = SU(2)$, even though both groups have the same Lie algebra.[63]

- The field $\mathbb{R}$ of real numbers is a group with respect to addition. If the fiber G is contractible, like $\mathbb{R}$ is, then all principal G-bundles are trivial if the base space is a CW complex.[64,65,66] In contrast, principal $U(1)$-bundles over T^n are classified by the second cohomology group $H^2(T^n; \mathbb{Z})$,[67] which is isomorphic to the direct sum of $\binom{n}{2}$ copies of $\mathbb{Z}$.[68] This shows that infinitely many non-isomorphic nontrivial principal G-bundles exist over T^n if $G = U(1)$, but none exist if $G = \mathbb{R}$, even though both groups have the same Lie algebra.

- Nontrivial principal G-bundles exist over S^2 if $G = U(1)$[69] but not if $G = \mathbb{R}$.

[63] As a manifold, $SU(2)$ is homeomorphic to S^3, and a nontrivial S^3 bundles over S^2 does exist (Steenrod (1951), section 26.3, page 135), but it is not a principal $SU(2)$-bundle. The fact that a nontrivial S^3 bundle over S^2 exists is related to the fact that a nontrivial principal $SO(4)$-bundle over S^2 exists (Boyer (2011), section 1).

[64] Freed (2015), proposition 12.26

[65] Kazukawa *et al* (2023) describes a nontrivial principal bundle with contractible fiber when the base space is not a CW complex (not paracompact).

[66] Nontrivial real line bundles over T^n exist for all $n \geq 1$ (the Möbius strip is a real line bundle over $T^1 = S^1$, with the real line depicted as a line segment), but they're not principal $\mathbb{R}$-bundles. They're associated with principal $\mathbb{Z}_2$-bundles instead.

[67] Article 35490

[68] Article 28539

[69] Section 9

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