Why the (Hilbert) Stress-Energy Tensor is Covariantly Conserved

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Abstract In general relativity, the same covariant conservation law for the stress-energy tensor of the matter fields can be derived using either of two different methods. One method relies on the equation of motion for the metric field, and the other method uses only the equation of motion for the matter fields. This article explains why both methods lead to the same conservation law.

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1 Introduction

The equation of motion (abbreviated **EoM** in this article) for the metric field in general relativity has the form

$$G^{ab}(x) \propto T^{ab}(x), \tag{1}$$

where the **Einstein tensor** G^{ab} depends only on the metric field, and the quantity T^{ab} on the right-hand side depends also on whatever other fields are present, hereafter called **matter fields**. The quantity T^{ab} in this equation is the (Hilbert) stress-energy tensor defined in equation (8), below. It is automatically both symmetric and gauge-invariant.

For any metric field whatsoever, whether or not it satisfies the EoM (1), the Einstein tensor G^{ab} automatically satisfies the identity

$$\nabla_a G^{ab}(x) = 0, \tag{2}$$

where ∇_a is the covariant derivative associated with the metric field. When combined with the EoM (1), this implies the **covariant conservation law**

$$\nabla_a T^{ab}(x) = 0. \tag{3}$$

When viewed this way, the conservation law is a condition that the matter fields' EoMs must satisfy in order to be consistent with the *metric* field's equation of motion.

The same conservation law (3) can also be derived a different way, without using the identity (2). It can be derived instead using only the matter fields' equations of motion, as long as the whole model is **generally covariant** as defined in the next section.

This article explains why these two seemingly different approaches – one relying on the equations of motion for the matter fields, and one not – both lead to the *same* conservation law (3) for the stress-energy tensor of the matter fields.

2 General covariance

In both approaches, the conservation law (3) is a consequence of **general covariance**. I'm not sure that name is always used consistently (most physics texts don't define it carefully enough), so I'll explain what I mean by it. Suppose that the model's equations of motion come from the action principle, with an action of the form

$$S = \int d^N x \, \sqrt{|\det g(x)|} \, L(x), \tag{4}$$

where N is the number of spacetime dimensions, and the lagrangian function $L(x)$ is (a coordinate representation of) a scalar field constructed from various other tensor fields. Tensor fields – including scalar fields – have coordinate-free definitions,[1] and article 00418 explains how any diffeomorphism[2] of the spacetime manifold in which they live induces a corresponding transformation of the fields. As in that article, I'll use the word **fieldomorphism** for this corresponding transformation of the fields.[3] General covariance means that the action is invariant under fieldomorphisms.[4]

This article only considers fieldomorphisms that are compactly supported in spacetime. This way, we can restrict the integral (4) to a compact domain (which contains the support of the fieldomorphisms under consideration) so that it is well-defined, and we can integrate-by-parts without generating boundary terms. The action is required to be invariant under all fieldomorphisms whose support is restricted to the interior of a region R, for all compact regions R.

[1] Article 09894 reviews the coordinate-free definitions of tensor fields.

[2] Article 93875 reviews the definition of **diffeomorphism**.

[3] This name is not standard.

[4] In the physics literature, general covariance is sometimes called diffeomorphism invariance, but mathematicians usually reserve the word *diffeomorphism* for its effect on the underlying smooth manifold, not for the corresponding effect on the fields.

3 Conservation from the metric field's EoM

This section shows how to derive the covariant conservation law (3) using the EoM for the metric field, *without* using any EoMs for the matter fields. This approach is based on these assumptions:

- The EoM for the metric field satsifies an action principle. In other words, it may be derived from an **action**, using the principle of stationary action.
- The action is the sum of two parts, each of which makes a non-zero contribution to the EoM: One part S_g involving only the metric field,[5] and one part S_m that involves both the matter fields and the metric field.
- The metric-only part S_g is invariant under fieldomorphisms (section 2).

Aside from these basic properties, the explicit form of the action doesn't matter. Most importantly, this approach does not assume anything about the EoMs for the matter fields.

To explore the consequences of those assumptions, consider an action of the form

$$S = S_g + S_m \tag{5}$$

with S_g and S_m as described above. According to the action principle, the EoM for the metric field is

$$\frac{\delta S}{\delta g_{ab}(x)} = 0. \tag{6}$$

This implies

$$G^{ab} \propto T^{ab}(x) \tag{7}$$

with

$$G^{ab}(x) \propto \frac{1}{\sqrt{g}}\frac{\delta S_g}{\delta g_{ab}(x)} \qquad\qquad T^{ab}(x) \equiv \frac{-2}{\sqrt{g}}\frac{\delta S_m}{\delta g_{ab}(x)}. \tag{8}$$

[5] The simplest option is the **Einstein-Hilbert action** for the metric field, which leads to general relativity.

Here, $\sqrt{g}$ is an abbreviation for the magnitude of the determinant of the metric field:[6]

$$\sqrt{g} \equiv \sqrt{|\det g|}.$$

We will see below that the general covariance of S_g implies

$$\nabla_a G^{ab} = 0. \tag{9}$$

Equation (9) is a consequence of the fieldomorphism invariance of the metric-only action S_g, regardless of any equations of motion. When combined with the EoM (7), this implies the conservation law (3).

The remaining task is to derive equation (9). For this, we only need the assumption that S_g is invariant under all fieldomorphisms. The effect of a generic fieldomorphism on the coordinates may be written

$$x^a \to x^a + \epsilon\theta^a(x), \tag{10}$$

where $\theta^a(x)$ is a smooth function, and ϵ is a fixed parameter that we will take to be infinitesimal in the following derivation. Article 71500 shows that when ϵ is infinitesimal, the effect of this fieldomorphism on the metric field is

$$\delta g_{ab}(x) = \Big(\nabla_a\,\theta_b(x) + \nabla_b\,\theta_a(x)\Big)\epsilon. \tag{11}$$

For an arbitrary variation of the metric field (not necessarily a fieldomorphism), we have the identity

$$\delta S_g = \int d^N x\ \frac{\delta S_g}{\delta g_{ab}(x)}\delta g_{ab}(x). \tag{12}$$

If we take the transformation $\delta g_{ab}(x)$ to be an infinitesimal fieldomorphism (11), then S_g is invariant, so we have

$$0 = \epsilon\int d^N x\ \frac{\delta S_g}{\delta g_{ab}(x)}\nabla_a\theta_b(x). \tag{13}$$

[6] The determinant of the metric field is never zero, so it doesn't cause any problems in the denominator of (8).

The symmetry in of g_{ab} was used to combine the two terms on the right-hand side of (11). Use the definition (8) to write this as

$$0 = \epsilon \int d^N x \ \sqrt{g}\, G^{ab}(x) \nabla_a \theta_b(x). \tag{14}$$

For the next step, we'll need a few properties of the covariant derivative ∇, namely[7]

$$\begin{aligned} \nabla_a \theta_b &= \partial_a \theta_b - \Gamma^c_{ab} \theta_c \\ \nabla_a G^{ab} &= \partial_a G^{ab} + \Gamma^a_{ac} G^{cb} + \Gamma^b_{ac} G^{ac} \\ \partial_a \sqrt{g} &= \sqrt{g}\, \Gamma^b_{ba}. \end{aligned}$$

Use integration-by-parts along with these identities to see that equation (14) implies

$$0 = \int d^N x \ \sqrt{g}\, \theta_b(x) \nabla_a G^{ab}(x). \tag{15}$$

The fact that equation (13) (and therefore equation (15)) holds for all compactly-supported smooth functions $\theta_b(x)$ implies equation (9).[8] When combined with the EoM (7), this implies the conservation law (3). This completes the derivation of the conservation law (3) from the assumptions that were listed at the beginning of this section. Most importantly, this derivation did *not* use any equations of motion for the matter fields. Instead, we deduced something (equation (3)) about how the matter fields must behave.

[7] These are derived in article 03519.

[8] In a little more detail: suppose that $f(x)$ is an unknown smooth function satisfying the condition that $\int_a^b dx\ \theta(x) f(x) = 0$ for all intervals $[a, b]$, for all smooth functions $\theta(x)$ that are zero in a neighborhood of a and b. Now choose any interval $[a, b]$ in which $f(x)$ is either positive or negative everywhere in that interval. Then no cancellations can occur if $\theta \geq 0$ everywhere in that interval, so the assumed condition cannot hold unless $f(x) = 0$ everywhere. The same idea works for higher-dimensional integrals, too.

4 Conservation from the matter fields' EoMs

This section shows that if the action for the matter fields is invariant under fieldomorphisms in the sense defined below, then the conservation law (3) holds. In this approach, the conservation law is a consequence of the EoMs for the matter fields alone. The EoM for the metric field (equation (6) or (7)) is not used. This approach is based on these assumptions:

- The EoMs for the matter fields collectively satsify an action principle. In other words, they may all be derived from a single action S_m, using the principle of stationary action. (The EoM for the metric field is not used, so it doesn't need to satisfy the action principle.)

- The action S_m is invariant under all fieldomorphisms, as long as the fieldomorphism is applied to the metric field, too.

Aside from these basic properties, the explicit form of the action doesn't matter. Most importantly, this approach does not assume anything about the EoM for the metric field. The metric field may simply be prescribed, and it may even be flat.[9]

Before continuing, I'll clarify what *general covariance* means when the metric is prescribed. In a context like general relativity,[10] where the metric field is one of the dynamic fields whose behavior is governed by the model's equations of motion (instead of being prescribed), *general covariance* implies that any fieldomorphism applied to any solution gives another solution. This fieldomorphism symmetry is a **gauge symmetry**, meaning that two solutions that can be obtained from each other by a fieldomorphism are regarded as being physically equivalent to each other. In contrast, in a context like generalized special relativity (article 33547) where the metric field is merely prescribed, the result of applying an arbitrary fieldomorphism to a solution is typically *not* another solution of the original equations of motion –

[9] The definition pf T^{ab} (equation (8)) requires specifying how the action depends on the metric field, but the result may be specialized to a flat metric after evaluating the variation with respect to g_{ab}.

[10] Here, I'm distinguishing between *general relativity* and *generalized special relativity.* This distinction is defined in article 33547.

unless we also apply that same fieldomorphism to the metric field. In this context, *general covariance* assumes that we also apply the same fieldomorphism to the metric field that we apply to the matter fields, even though the metric field is one of the model's prescribed inputs.

Let $\phi_1, \phi_2, ...$ denote the list of matter fields, and suppose that the matter action S_m is invariant under fieldomorphisms in the sense described above. The effect of a fieldomorphism on the metric field is given by equation (11). Exactly how the fieldomorphism affects the matter fields isn't important in this derivation, so we can just write the effect as $\delta\phi_n$. For an arbitrary variation of the fields (not necessarily a fieldomorphism), including the metric field, we have the identity

$$\delta S_m = \int d^N x \ \left(\sum_n \frac{\delta S_m}{\delta \phi_n(x)} \delta\phi_n(x) + \frac{\delta S_m}{\delta g_{ab}(x)} \delta g_{ab}(x) \right).$$

If we take the transformation to be a fieldomorphism, then the assumption that S_m is invariant gives

$$0 = \int d^N x \ \left(\sum_n \frac{\delta S_m}{\delta \phi_n(x)} \delta\phi_n(x) + 2\epsilon \frac{\delta S_m}{\delta g_{ab}(x)} \nabla_a \theta_b(x) \right). \tag{16}$$

This is analogous to the result (13), which is a consequence of the fieldomorphism invariance of S_g, but now we have an extra term because the action S_m involves matter fields fields as well as the metric field. If the matter fields satisfy their own EoMs, namely

$$\frac{\delta S_m}{\delta \phi_n(x)} = 0,$$

then equation (16) reduces to

$$0 = \int d^N x \ \frac{\delta S_m}{\delta g_{ab}(x)} \nabla_a \theta_b(x). \tag{17}$$

This is just like equation (13), but with S_m in place of S_g, so we get the same result (9) but with T^{ab} in place of G^{ab}. This gives the conservation law (3), but here we derived it without using any EoM for the metric field. We used only general covariance of the matter part of the action combined with the EoM for the matter fields.

5 Why both approaches give the same result

The preceding sections derived the same conservation law (3) using two different methods. The difference between the two methods is that one method uses the EoM for the metric field, while the other method uses only the EoMs for the matter fields.

Now we can understand why both approaches give the same result in general relativity. In general relativity, the action has the form $S_g + S_m$, and S_g and S_m are each separately invariant under fieldomorphisms. The term S_g involves only the metric field, so the fieldomorphism invariance of S_g gives the identity (2), and then consistency with the metric field's equation of motion (7) requires that the matter fields satisfy the conservation law (3). The term S_m involves both matter fields and the metric field, so the fieldomorphism invariance of S_m gives the identity (16), which reduces to the conservation law (3) when the matter fields satisfy their equations of motion. This explains why the same conservation law can be derived either way, at least if explaining "why" means finding a short list of basic conditions that make the coincidence inevitable.

6 References

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