Homotopy Sets

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Abstract If X and Y are topological spaces and f is a map from X to Y, then the **homotopy class** $[f]$ is the set of all maps from X to Y that are **homotopic** to f, which roughly means they can be continuously morphed to f. The **(free) homotopy set** $[X, Y]$ is the set whose elements are homotopy classes of maps from X to Y. The **based homotopy set** is defined similarly, but using only homotopies that preserve a designated basepoint in X and in Y. The study of homotopy sets is a prominent part of the study of topology. Homotopy groups (article 61813) and cohomology groups (article 28539) may both be expressed as special families of homotopy sets equipped with a natural group structure. This article gathers some results about homotopy sets, with special attention given to the example $[T^3, SU(k)]$ where T^3 is a 3-torus and $SU(k)$ is a special unitary group.

Contents

1 Conventions and notation

- *Map* means continuous map, and *function* means continuous function.
- Each topological space is assumed to be homeomorphic to a CW complex.[1,2]
- S^n is an n-dimensional sphere, also called an ***n*-sphere**.
- T^n is an n-dimensional torus (the cartesian product of n circles), also called an ***n*-torus**.
- $\mathbb{Z}$ is the integers, $\mathbb{Z}_k$ is the integers modulo k, and $\mathbb{R}$ is the real numbers.
- Section 2 will introduce $[X,Y]$, the set of free homotopy classes of maps from one topological space X to another topological space Y.
- Section 3 will introduce $[X,Y]_0$, the set of based homotopy classes of maps.
- $\pi_j(X)$ is the jth homotopy group of a topological space X.[3]
- A topological space X is called ***n*-connected** if $\pi_j(X)$ is trivial for all $j \leq n$.[3] In particular, **1-connected** means $\pi_0(X)$ and $\pi_1(X)$ are both trivial. The word **connected** by itself is an abbreviation for 0-connected.
- $H^j(X;\mathbb{Z})$ is the jth integer cohomology group of a topological space X.[4]
- $U(k)$ and $SU(k)$ are the unitary and special unitary groups.
- A group or homotopy set is called **trivial** if it has only one element.
- If G and H are groups, then $G \simeq H$ means G and H are isomorphic to each other. If X and Y are topological spaces, then $X \simeq Y$ means X and Y are homeomorphic to each other.

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[1] Article 93875 defines **CW complex**.

[2] Every smooth manifold is homeomorphic to a CW complex (article 93875).

[3] Article 61813

[4] Article 28539

2 Free homotopy sets

Article 61813 introduces the concept of **homotopy**. Intuitively, a homotopy from one map $f : X \to Y$ to another map $g : X \to Y$ is a continuous deformation from $f(X)$ to $g(X)$ within Y. If a homotopy exists between f and g, then f and g are said to be **homotopic** to each other. For any given $f : X \to Y$, the set $[f]$ of all maps homotopic to f is called a **homotopy class**. A map $f : X \to Y$ is called **nullhomotopic** if it's homotopic to a **constant map**, which is a map that sends all of X to a single point of Y.

When X and Y are topological spaces, $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. Each element of $[X, Y]$ is a homotopy class $[f]$ of maps $f : X \to Y$. Section 1 in Čadek *et al* (2014) says,

> A central theme in algebraic topology is to understand, for given topological spaces X and Y, the set $[X, Y]$ of homotopy classes of maps from X to Y. Many of the celebrated results throughout the history of topology can be cast as information about $[X, Y]$ for particular spaces X and Y.

This article is motivated by applications to the study of principal G-bundles,[5] which are important in quantum field theory with gauge invariance.

[5] Article 33600

3 Based homotopy sets

If we choose a point $x_0 \in X$ and a point $y_0 \in Y$, then a homotopy that preserves the relationship $x_0 \to y_0$ throughout the deformation process is called a **based homotopy**. The points x_0, y_0 are called **basepoints**. Each element of the **based** (or **pointed**) **homotopy set** $[X, Y]_0$ is an equivalence class $[f]_0$ of maps using based homotopy as the equivalence relation. If Y is connected, then the homotopy group $\pi_n(Y)$ may be defined as the set $[S^n, Y]_0$ equipped with an appropriate group operation:[6]

$$\pi_n(Y) = [S^n, Y]_0. \tag{1}$$

The set $[X, Y]$ is sometimes called a **free homotopy set**[7] to distinguish it from the based homotopy set $[X, Y]_0$. The class $[f]_0$ is a subset of the class $[f]$, because $[f]$ includes all maps that are homotopic to f, whether or not they respect the basepoints.[8]

The notation $[X, Y]_0$ is common,[9,10] but sources that don't use free homotopy sets often use $[X, Y]$ to denote the based homotopy set.[11,12]

[6] Whitehead (1978), section III.5, text above corollary 5.23; May (2007), section 9.1

[7] Matumoto *et al* (1984), section 1

[8] Article 61813 describes an example where $[f]_0$ and $[f]$ differ, and `https://math.stackexchange.com/questions/2118574/` describes another one.

[9] Davis and Kirk (2001), section 6.9; Mimura and Toda (1991), section 4.1

[10] Hatcher (2001) writes $\langle X, Y\rangle$ instead of $[X, Y]_0$ (text above proposition 4.22).

[11] May (2007), section 8.1; May and Ponto (2012), beginning of section 1.4; Arkowitz (2011), page viii

[12] The beginning of section 7.1 in Cohen (2023) says, "In this chapter, unless otherwise specified, we will assume that all spaces are connected and come equipped with a basepoint. When we write $[X, Y]$ we mean homotopy classes of basepoint preserving maps $X \to Y$."

4 When free and based homotopy sets are equal

Suppose that X and Y are both connected CW complexes.[13] The free homotopy set $[X,Y]$ and the based homotopy set $[X,Y]_0$ are not always equal to each other, but they are in these cases, among others:

- If Y is 1-connected,[14] then $[X,Y] = [X,Y]_0$.[15]
- If Y is a connected **H-space**, then $[X,Y] = [X,Y]_0$.[16] Every topological group (which includes every Lie group) is an H-space.[17]

If X and Y are connected, then $[X,Y]$ is equal to $[X,Y]_0$ modulo an appropriate action of $\pi_1(Y)$,[18,19] so they're equal to each other whenever that action is trivial.

A space Y is called ***n*-simple** if the action of $\pi_1(Y)$ on $\pi_n(Y)$ is trivial.[20] A space Y is 1-simple if and only if $\pi_1(Y)$ is abelian.[21] Using equation (1), that result may also be expressed this way:

- If $\pi_1(Y)$ is abelian, then $[S^1,Y] = [S^1,Y]_0$.

[13]The sources cited in footnotes 15 and 16 assume that the spaces are *compactly generated*. Every CW complex has that property (article 93875). They also assume that the basepoints are *nondegenerate* (Davis and Kirk (2001), definition 6.31; May and Ponto (2012), beginning of section 1.1). Again, every CW complex has that property (Frankland (2013), example 1.2).

[14]Section 1 defines *1-connected*.

[15]Davis and Kirk (2001), corollary 6.59

[16]May and Ponto (2012), proposition 1.4.3 (and comment 0.0.3 on page xxii for the *connected* premise)

[17]Whitehead (1978), section III.4, page 119; Mimura and Toda (1991), section 2.4, page 69

[18]May and Ponto (2012), lemma 1.4.2; Davis and Kirk (2001), theorem 6.57; Bott and Tu (1982), proposition 17.6.1 (for $X = S^n$)

[19]Each element of $\pi_1(Y)$ is represented by a closed path in Y. The action of $\pi_1(Y)$ on $[X,Y]_0$ transports Y's basepoint around that closed path (Davis and Kirk (2001), definition 6.55 and the text above theorem 6.57).

[20]Arkowitz (2011), definition 5.5.7; Davis and Kirk (2001), definition 6.61

[21]Arkowitz (2011), text below definition 5.5.7; Davis and Kirk (2001), exercise 113

5 n-equivalence

Consider two CW complexes X and Y. Any map $f : X \to Y$ induces maps

$$f_* : [A, X]_0 \to [A, Y]_0 \tag{2}$$

for all A, because each map $A \to X$ may be composed with f to get a map $A \to Y$. For the same reason, $f : X \to Y$ induces maps

$$f_* : \pi_j(X, x) \to \pi_j(Y, f(x)) \tag{3}$$

of homotopy groups with the indicated basepoints. The induced maps (2)-(3) are not always bijective,[22] but if A is a CW complex and n is any positive integer, these two conditions are equivalent to each other:[23]

- the map (2) is bijective when $\dim A < n$ and surjective when $\dim A = n$,
- the map (3) is bijective when $j < n$ and surjective[24] when $j = n$. In other words, f is an ***n*-equivalence**.[25]

For most maps $X \to Y$, neither of these conditions holds, but if either one of them does hold, then so does the other one.

[22] A map $A \to B$ is called **bijective** if each element of B is the image of exactly one element of A.

[23] Matumoto *et al* (1984), theorem 2; May (2007), chapter 10, section 3; Arkowitz (2011), definition 2.4.4 and proposition 2.4.6; Whitehead (1978), chapter 4, theorem 7.16 (recalled in Whitehead (1978), chapter 5, beginning of section 3)

[24] A map $A \to B$ is called **surjective** if each element of B is the image of one or more elements of A.

[25] May (2007), chapter 9, section 6

6 n-equivalence for large n

If the induced maps (3) are isomorphisms for all j, then f is called a **weak homotopy equivalence**.[26] If X and Y are CW complexes, then a weak homotopy equivalence is a homotopy equivalence,[27] so in this case the induced map (2) is bijective for all CW complexes A.[28]

A similar result is true for finite n if X and Y are both CW complexes with dimension less than n: in this case, an n-equivalence between X and Y is a homotopy equivalence,[29] which again implies that (2) is bijective for all A.

[26] Davis and Kirk (2001), definition 7.30; Hatcher (2001), §4.1, p 352

[27] Maxim (2013), theorem 5.4.1; Mitchell (1997), theorem 6.4 and the text above it; May (2007), the beginnings of chapter 10 and of section 6 in chapter 9

[28] Davis and Kirk (2001), theorem 7.32; Hatcher (2001), proposition 4.22

[29] May (2007), chapter 10, section 3

7 $[M, S^n]$ when M is n-dimensional

If M is a closed, compact, connected, and oriented n-dimensional manifold, then $[M, S^n] \simeq \mathbb{Z}$, where the integer assigned to a map $M \to S^n$ is called the **degree** of the map.[30] When $M = S^n$, this may also be written $\pi_n(S^n) \simeq \mathbb{Z}$.

To construct an example of a map $M \to S^n$ that is nullhomotopic, choose any n-dimensional ball U in M, and choose any point p in S^n. A map $f : M \to S^n$ with $f : U \to S^n \setminus p$ bijective and $f(M \setminus U) = p$ is not nullhomotopic.[31,32]

A map $X \to Y$ is called **surjective** if every element of Y is the image of at least one element of X. A map $M \to S^n$ with nonzero degree is surjective,[33] but it's not a *covering map* in the sense defined in article 61813. A covering map $X \to Y$ assigns k points of X to each point of Y, with the same k everywhere. Any covering map from a sphere S^n to itself necessarily has $|k| = 1$. A generic continuous map $S^n \to S^n$ may have any degree, but such a map cannot be k-to-1 with the same k everywhere. To construct an example of a map $S^n \to S^n$ with degree 2, think of S^n as the set of points $(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$ with $\sum_j x_j^2 = 1$. Write $x_0 = \cos\theta$ and $x_j = \hat{x}_j \sin\theta$ for $j \in \{1, ..., n\}$ with $\sum_j \hat{x}_j^2 = 1$ and $0 \leq \theta \leq \pi$. Then the map $S^n \to S^n$ defined by $(\theta, \hat{x}_j) \to (2\theta, \hat{x}_j)$ has degree 2. This map is 2-to-1 almost everywhere, but not where θ is an integer multiple of $\pi/2$, because it sends the whole equator ($x_0 = 0$) of the first S^n to a single point ($x_0 = -1$) of the second S^n.

[30] Kosinski (1993), chapter IV, corollary 5.8

[31] This is a special case of the construction described in the proof of theorem 1.10 in Hirsch (1976), chapter 5. That construction gives a map $M \to S^n$ of degree m for each $m \in \mathbb{Z}$. (Beware of what I assume is a typo in that source: "$i = 1, ..., n$" should presumably be "$i = 1, ..., m$.")

[32] $X \setminus Y$ denotes the part of X that remains after deleting Y.

[33] Hirsch (1976), chapter 5, text below theorem 1.6

8 The wedge sum and the smash product

An n-torus, denoted T^n, is a cartesian product of n circles. In particular, $T^2 = S^1 \times S^1$. Section 10 will determine $[T^2, M]$ when M is 1-connected. This section introduces some of the ingredients in that calculation.

Consider two topological spaces X and Y with designated basepoints $x_0 \in X$ and $y_0 \in Y$. Their **wedge sum**, denoted $X \vee Y$, is the subset of their cartesian product $X \times Y$ defined by the union of $X \times y_0$ and $x_0 \times Y$.[34,35] Given two maps $f : X \to M$ and $g : Y \to M$, a map $\{f, g\} : X \vee Y \to M$ can be defined for which the induced map

$$[X, M]_0 \times [Y, M]_0 \to [X \vee Y, M]_0 \tag{4}$$

given by $([f], [g]) \mapsto [\{f, g\}]$ is bijective.[36,37]

The **smash product** of two spaces X and Y, denoted $X \wedge Y$, is defined by starting with $X \times Y$ and then collapsing a wedge $X \vee Y \subset X \times Y$ to a single point:[34]

$$X \wedge Y \equiv \frac{X \times Y}{X \vee Y}.$$

Section 9 will use easy examples to illustrate these things.

[34] Hatcher (2001), chapter 0, page 10

[35] This is not related to the wedge product $a \wedge b$ of vectors a and b (article 81674).

[36] Arkowitz (2011), corollary 1.3.7

[37] Arkowitz (2011) writes $[X, Y]$ for the based homotopy set, which is denoted $[X, Y]_0$ here.

9 Examples

The wedge sum $S^1 \vee S^1$ is a pair of circles that intersect each other at a single point, so it has the same topology as the symbol ∞. The two-dimensional real projective space $\mathbb{R}\mathrm{P}^2$ has fundamental group[38]

$$[S^1, \mathbb{R}\mathrm{P}^2]_0 = \pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2.$$

Use this in the bijection (4) to get

$$[S^1 \vee S^1, \mathbb{R}\mathrm{P}^2]_0 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2,$$

which shows the number of classes of based homotopies from $S^1 \vee S^1$ to $\mathbb{R}\mathrm{P}^2$ is four.

The space $S^1 \vee S^1$ is not a manifold, because the point where the circles intersect does not have an open neighborhood homeomorphic to any euclidean space, but it is a CW complex: it is made from two 1-cells whose endpoints meet at a single 0-cell.

The torus T^2 may also be given the structure of a CW complex in which $S^1 \vee S^1$ is a subcomplex: it is made from a subset of the cells from which T^2 is made. To deduce this, think of the torus $T^2 = S^1 \times S^1$ as a rectangle with opposite sides identified. Let U denote the interior of this rectangle. This is a 2-cell. The remainder $T^2 \setminus U$ is the boundary of the rectangle, and identifying opposite sides makes it a wedge sum of two circles: $T^2 \setminus U \simeq S^1 \vee S^1$. Altogether, this represents the torus as the union of a 2-cell U and a 1-dimensional subcomplex $S^1 \vee S^1$.

The smash product of two circles is a 2-sphere. We can deduce this by using the representation in the preceding paragraph. The smash product $S^1 \wedge S^1 = T^2/(S^1 \vee S^1)$ is defined by treating the rectangle's boundary $S^1 \vee S^1$ as a single point. The rectangle's interior is homeomorphic to the interior of a disk, so $S^1 \wedge S^1$ is topologically the same as treating a disk's boundary as a single point. This gives a sphere S^2.

[38] Article 61813

10 $[T^2, M]$ when M is 1-connected

This section shows that if a CW complex M is 1-connected, then the set $\pi_2(M)$ is the same as $[T^2, M]$.[39]

If a CW complex A is a subcomplex of a CW complex X, then the inclusion $A \to X$ qualifies as something called a **cofibration**.[40] If an inclusion $A \to X$ is a cofibration, then this sequence is exact:[41]

$$[\text{point}, M] \to [X/A, M] \to [X, M] \to [A, M],$$

where the second map is the pullback of the projection $X \to X/A$ and the third one is the pullback of the inclusion $A \to X$. Applying this to the case described in section 9 gives an exact sequence

$$(\text{one-element set}) \to [S^2, M] \to [T^2, M] \to [S^1 \vee S^1, M]. \tag{5}$$

Now suppose that M is 1-connected. In this case, the based homotopy set $[X, M]_0$ is the same as the free homotopy set $[X, M]$, and the homotopy groups $\pi_j(M)$ are the same (as sets) as the free homotopy sets $[S^j, M]$.[42] Use these facts in (5), together with the bijection (4) to get an exact sequence

$$(\text{one-element set}) \to \pi_2(M) \to [T^2, M] \to (\text{one-element set}). \tag{6}$$

The exactness of (6) implies that the set $\pi_2(M)$ is the same as $[T^2, M]$, as claimed at the beginning of this section.

[39] This is a special case of a result derived in https://mathoverflow.net/questions/234367/.

[40] Arkowitz (2011), proposition 3.2.4

[41] A sequence of maps is called **exact** if the image of each map equals the kernel of the next map (article 29682). Exactness of the part of the sequence going in and out of $[X, M]$ is theorem 6.30 in Davis and Kirk (2001). Exactness of the part that goes in and out of $[X/A, M]$ follows from teh fact that the quotient map $X \to X/A$ is surjective, so if $X \to X/A \to M$ is nullhomotopic, then $X/A \to M$ is, too. This shows that the kernel of $[X/A, M] \to [X, M]$ consists of nullhomotopic maps, which is precisely the image of $[\text{point}, M] \to [X/A, M]$.

[42] Section 3

11 A warning about concatenated homotopies

Sections 12-12 will show that $[T^3, SU(k)]$ is nontrivial for $k \geq 2$. The hardest step is showing that $[T^3, SU(3)]$ is nontrivial. We might be tempted to show this using maps of the form $T^3 \to S^3 \to SU(3)$, because we know that $[T^3, S^3]$ and $[S^3, SU(3)]$ are both nontrivial.[43] This doesn't automatically imply that $[T^3, SU(3)]$ is also nontrivial, though, because a map $X \to Z$ of the form $X \to Y \to Z$ may be homotopic to a constant map even if the constituent maps $X \to Y$ and $Y \to Z$ are not. This section describes examples of that phenomenon.

For one example, use $X = Y = S^1$ and $Z = \mathbb{R}\mathrm{P}^2$. Represent S^1 as the unit circle in the complex plane, and take the map $X \to Y$ to be the double-covering defined by $e^{i\theta} \mapsto e^{i\,2\theta}$. To describe the map $Y \to Z$, use a pair of coordinates (a, b) to denote points of $\mathbb{R}^2$, and think of $\mathbb{R}\mathrm{P}^2$ as a disk in $\mathbb{R}^2$ of radius π with opposite points of its boundary identified with each other. Define a map $Y \to Z$ by $e^{i\theta} \mapsto (\theta, 0)$ for $-\pi < \theta \leq \pi$. Then the maps $X \to Y$ and $Y \to Z$ are both homotopically nontrivial (neither one can be continuously morphed to a constant map), but their composition can be continuously morphed to a constant map.[44]

The same phenomenon occurs with $Z = \mathbb{R}\mathrm{P}^3$ in place of $Z = \mathbb{R}\mathrm{P}^2$, which shows that it can occur even when X, Y, Z are all orientable manifolds.

The phenomenon can still occur if X, Y, Z are all 1-connected orientable manifolds. In particular, it occurs when $X = S^4$, $Y = S^3$, and $Z = SU(3)$. In this case, all compositions $X \to Y \to Z$ are homotopically trivial because $\pi_4(SU(3)) = 0$, even though both individual maps in the composition may be homotopically nontrivial because $\pi_4(S^3) = \mathbb{Z}_2$ and $\pi_3(SU(3)) = \mathbb{Z}$.[45]

[43] Section 7 says $[T^n, S^n] \simeq \mathbb{Z}$, and article 92035 says $\pi_3(SU(3)) \simeq \mathbb{Z}$.

[44] In the composition $X \to Y \to Z$, the image of X is a loop that starts at $(-\pi, 0)$, travels through the disk to $(\pi, 0)$, which is equivalent to $(-\pi, 0)$, and then travels again from $(-\pi, 0)$ through the disk to $(\pi, 0)$. This map $X \to Z$ can be morphed to on that starts at $(-\pi, 0)$, travels through the disk to $(\pi \cos \epsilon, \pi \sin \epsilon)$, which is equivalent to $(-\pi \cos \epsilon, -\pi \sin \epsilon)$, and then travels from there through the disk to $(\pi, 0)$ so that the image of X is still a closed loop. By continuously morphing ϵ from 0 to π, we can morph the original map $X \to Z$ to a constant map.

[45] Articles 61813 and 92035

12 Nontriviality of $[T^3, SU(k)]$

The fact that $[T^3, SU(2)]$ is nontrivial follows from the homeomorphism[46] $SU(2) \simeq S^3$ combined with the fact that $[M, S^3] \simeq \mathbb{Z}$ for any closed 3-dimensional manifold M.[47]

To show that $[T^3, SU(k)]$ is nontrivial for all $k \geq 2$, we can use another result that relates the case k to the case $k+1$. If $E \to B$ is a fiber bundle with fiber F, then this sequence of induced maps between homotopy groups is exact:[48]

$$\cdots \to \pi_{j+1}(B) \to \pi_j(F) \to \pi_j(E) \to \pi_j(B) \to \cdots \to \pi_1(B). \tag{7}$$

This is called the **homotopy sequence of the bundle**.[49] Set[50]

$$F = SU(k) \qquad E = SU(k+1) \qquad B = SU(k+1)/SU(k)$$

and use the relationships[51]

$$SU(k+1)/SU(k) \simeq S^{2k+1} \text{ for } k \geq 2 \qquad\qquad \pi_j(S^{2k+1}) = 0 \text{ for } j \leq 2k$$

to conclude that $\pi_j(F) \to \pi_j(E)$ is bijective for $j < 2k$ and surjective for $j = 2k$, and then use section 5 to conclude that $[T^n, SU(k)]$ and $[T^n, SU(k+1)]$ have the same number of elements if $n < 2k$. Set $n = 3$ and $k = 2$ to deduce that $[T^3, SU(2)]$ and $[T^3, SU(3)]$ have the same number of elements, which implies that $[T^3, SU(4)]$ also has the same number of elements, and so on. We already know that $[T^3, SU(2)]$ is nontrivial,[52] so this shows that $[T^3, SU(k)]$ is nontrivial for every $k \geq 2$.

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[46] Article 92035

[47] Section 7

[48] Davis and Kirk (2001), lemma 6.54

[49] Steenrod (1951), section 17.3

[50] Article 35490 shows that a fiber bundle with these ingredients exists.

[51] Mimura and Toda (1991), chapter 1, theorem 2.10 and page 68 (for the first relationship) article 61813 (for the second relationship)

[52] Section 12

13 Nontriviality of $[T^3, SU(3)]$: cross-check

As a cross-check, this section uses a different method to rederive the fact that $[T^3, SU(3)]$ is nontrivial.

The Lie group $SU(3)$ and the product $S^3 \times S^5$ are both 8-dimensional manifolds. They are not homeomorphic to each other,[53] but they do have the same homology groups,[54] so they are more similar to each other than the number of dimensions alone would suggest. This section shows that $[T^3, S^3 \times S^5]$ is nontrivial and then uses that result to deduce that $[T^3, SU(3)]$ must also be nontrivial.

To show that $[T^3, S^3 \times S^5]$ is nontrivial, use the general relationship[55,56]

$$[X, A \times B] \simeq [X, A] \times [X, B]. \tag{8}$$

Using the relationship[57]

$$[T^3, S^3] \simeq \mathbb{Z}$$

in (8) shows that $[T^3, S^3 \times S^5]$ has at least as many elements as $\mathbb{Z}$.

The next goal is to relate $[T^3, S^3 \times S^5]$ to $[T^3, SU(3)]$. This will be done by establishing the existence of a map $f : S^3 \times S^5 \to SU(3)$ for which the induced homomorphisms $\pi_j(S^3 \times S^5) \to \pi_j(SU(3))$ are bijective for $j \leq 3$ and surjective for $j = 4$. This is automatic for $j \in \{1, 2, 4\}$,[58] but the case $j = 3$ depends on the map f.[59] If such a map does exist, then the lemma reviewed in section 5 implies that the induced map $[T^3, S^3 \times S^5] \to [T^3, SU(3)]$ is bijective, and combining this with the previous paragraph shows that $[T^3, SU(3)]$ is nontrivial.

The remaining task is to establish the existence of a map $f : S^3 \times S^5 \to SU(3)$ for which the induced homomorphism $\pi_3(S^3 \times S^5) \to \pi_3(SU(3))$ is bijective. The man-

[53] To confirm this, use $\pi_4(S^3 \times S^5) \simeq \mathbb{Z}_2$ (article 61813) and $\pi_4(SU(3)) = 0$ (article 92035).

[54] Article 92035

[55] Arkowitz (2011), corollary 1.3.7. That book uses the notation $[X, Y]$ for what this article calls $[X, Y]_0$, but based and unbased homotopy sets are the same when Y is 1-connected (article 61813).

[56] This relationship is also used in the proof of equation (7) in Wang (2021a).

[57] Section 7

[58] $\pi_j(S^3 \times S^5)$ and $\pi_j(SU(3))$ are both trivial for $j \in \{1, 2\}$, and $\pi_j(SU(3))$ is trivial for $j = 4$.

[59] $\pi_3(S^3 \times S^5)$ and $\pi_3(SU(3))$ are both isomorphic to $\mathbb{Z}$, and a homomorphism $\mathbb{Z} \to \mathbb{Z}$ may or may not be bijective.

ifolds $SU(2)$ and S^3 are homeomorphic to each other,[60] so we can think of $S^3 \times S^5$ as the total space of a (trivial) principal $SU(2)$-bundle over S^5. The quotient manifold $SU(3)/SU(2)$ is homeomorphic to S^5,[60] so we can also think of $SU(3)$ as the total space of a (nontrivial) principal $SU(2)$-bundle over S^5. Up to isomorphism, these are the only two principal $SU(2)$-bundles over S^5.[61,62] To construct a map f with the desired property, start with the nontrivial bundle $SU(3) \to S^5$, choose a map $g : S^5 \to S^5$ of degree 2,[63] and consider the pullback of the nontrivial bundle by this map.[64,65] Principal $SU(2)$-bundles over S^5 are classified by $\pi_4(SU(2)) \simeq \mathbb{Z}_2$, so the fact that g has degree 2 implies that the resulting bundle must be the trivial bundle with total space $S^3 \times S^5$. This relationship between the two bundles provides a map $f : S^3 \times S^5 \to SU(3)$ from one total space to the other.[66]

To show that this map f induces a bijection $\pi_3(S^3 \times S^5) \to \pi_3(SU(3))$, use this general property of pullback bundles: if the original bundle is $E \to B$ and the map $f : B' \to B$ gives the pullback bundle $E' \to B'$, then this induced sequence of homotopy groups is exact:[67]

$$\cdots \to \pi_{j+1}(B) \to \pi_j(E') \to \pi_j(E) \oplus \pi_j(B') \to \pi_j(B) \to \cdots .$$

Set $B = B' = S^5$, $E' = S^3 \times S^5$, and $E = SU(3)$ and use $\pi_j(S^5) = 0$ for $j \leq 4$ to infer $\pi_j(E') \simeq \pi_j(E)$ for $j \leq 3$. This shows that f has the desired property, which is the last ingredient we needed to complete the proof that $[T^3, SU(3)]$ is nontrivial.

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[60] Section 12

[61] This follows from the fact that $\pi_4(SU(2)) \simeq \pi_4(S^3) \simeq \mathbb{Z}_2$ has exactly two elements.

[62] A third S^3 bundle over S^5 exists (Steenrod (1951), section 26.9; Wang (2021a)), but it's not a principal bundle. It's a nontrivial fiber bundle that admits a section (Wang (2021b), theorem 1.1).

[63] The identity $\pi_j(S^j) \simeq \mathbb{Z}$ implies that such a map exists (section 7).

[64] Lafont and Neofytidis (2019), in the proof of lemma 4.2

[65] Article 35490 defines *pullback bundle*.

[66] Hatcher (2001), section 3.H, pages 332-333; Whitehead (1978), chapter 1, text above corollary 7.22

[67] Whitehead (1978), chapter 5, top of page 254

14 Homotopy sets and cohomology

Section 3 reviewed how homotopy groups may be expressed in terms of homotopy sets. This section reviews how cohomology groups may be expressed in terms of homotopy sets.[68]

Choose a group G and a positive integer n. A topological space X with the property

$$\pi_k(X) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise} \end{cases}$$

is called an **Eilenberg-MacLane space**,[69] denoted $K(G, n)$. A CW complex satisfying this condition exists for each group G if $n = 1$ and for each abelian group if $n \geq 2$.[70,71] It is determined uniquely by G and n up to homotopy equivalence.[72]

When G is abelian, Eilenberg-MacLane spaces are important because of their relationship to cohomology groups. The homotopy set $[X, K(G, n)]$ can be given the structure of an abelian group in a natural way.[73] If X is a CW complex, G is an abelian group, and $K(G, n)$ is an Eilenberg-MacLane space, then the group $[X, K(G, n)]$ and the cohomology group $H^n(X; G)$ are isomorphic to each other:[74,75]

$$[X, K(G, n)] \simeq H^n(X; G). \tag{9}$$

The isomorphism is called the **Eilenberg-MacLane map**.[76]

[68] Article 28539 includes a preview of cohomology groups.

[69] Cohen (2023), definition 4.4; Davis and Kirk (2001), definition 7.19

[70] Davis and Kirk (2001), theorem 7.20; Hatcher (2001), section 4.2, page 365; Cohen (2023), theorem 7.19 and the text above theorem 7.23

[71] Similarly, for each abelian group G and each integer $n \geq 2$, a 1-connected CW complex X exists whose homology groups $H_k(X; \mathbb{Z})$ are G for $k = n$ and zero otherwise (Arkowitz (2011), lemma 2.5.2 and definition 2.5.3). Such an X is called a **Moore space**, denoted $M(G, n)$.

[72] Hatcher (2001), proposition 4.30

[73] Arkowitz (2011), text above definition 2.5.10

[74] Davis and Kirk (2001), theorem 7.22, previewed on page 168; Arkowitz (2011), end of section 2.1, definition 2.5.10, remark 2.5.11, and beginning of section 5.1

[75] This is also true using a based homotopy set $[X, K(G, n)]_0$ in place of the free homotopy set $[X, K(G, n)]$ (Hatcher (2001), theorem 4.57 and section 4.3, page 394).

[76] Husemöller *et al* (2008), chapter 9, theorem 6.3

15 Example

The circle S^1 is a $K(\mathbb{Z},1)$.[77,78] Use this in equation (9) to get

$$[M,U(1)] \simeq [M,S^1] \simeq [M,K(\mathbb{Z},1)] \simeq H^1(M;\mathbb{Z}).$$

This says that homotopy classes of maps $M \to U(1)$ correspond one-to-one with elements of the first cohomology group $H^1(M;\mathbb{Z})$.

[77] Article 61813

[78] Usually, a $K(G,n)$ is not a finite-dimensional manifold.

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