States With and Without State-Vectors

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Abstract A **normalized positive linear functional** – more concisely called a **state** – is the mathematical structure that quantum theory uses to assign probabilities to the various possible outcomes of a measurement, based on whatever we already know about the current status of the physical system. This article introduces the mathematical concept of a state and its relationship to state-vectors. For motivation, Gleason's theorem and its relatives are briefly reviewed. A general form of the Cauchy-Schwarz inequality is derived.

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1 Motivation

In quantum theory, projection operators (article 74088) are used to represent the possible outcomes of a measurement. If every measurable quantity had a well-defined value whether or not it were actually measured, then we should be able to assign a boolean value $\in \{0, 1\}$ to every projection operator, indicating whether that outcome would be realized if the relevant measurement were done. However, the fact that most projection operators do not commute with each other makes such an assignment mathematically impossible. Specifically, the **Kochen-Specker theorem**[1] says that such an assignment is impossible for all of the projections operators on a three-dimensional Hilbert space,[2] which is a subset of the projection operators used to represent the possible measurement outcomes for the various observables in any nontrivial model.

The next best thing is to assign *probabilities* to the possible outcomes, which is what quantum theory does. The standard assumption is that the set of projection operators representing all possible outcomes of all possible measurements coincides with the set of *all* projection operators in some noncommutative **von Neumann algebra**.[3] If we insist on being able to assign probabilities to all of the projection operators in a consistent way,[4,5] then the theorems reviewed below apply.

[1] A concise version of the proof is shown in Conway and Kochen (2009) and in endnote 4 of Conway and Kochen (2006).

[2] Peres (1990) presents an even easier proof (attributed to Mermin) in the context of a *four*-diensional Hilbert space, but with the help of an extra assumption that the Kochen-Specker theorem does not need.

[3] Article 74088 introduces the concept of a von Neumann algebra. One example is the algebra of all linear operators on a given Hilbert space.

[4] Here, *consistent* means satisfying conditions (1)-(3), below. If we also try to make the assignment consistent with naïve conditions extrapolated from everyday experience, then we run into problems. This is the significance of **Bell inequalities**.

[5] This is probably more than we really need, for two reasons. First, many of the things that the standard assumption formally designates as "observables" might not actually be measurable, given the limited resources available in the physical universe. This is analyzed in Omnès (1994), chapter 7, section 8, pages 308-309, which concludes "Some observables cannot be measured, even as a matter of principle." Second, not all observables are *actually* measured, even if they *could* be measured. For making predictions, we really only need to assign probabilities to the possible outcomes of observables that will actually be measured.

Mathematically, assigning a probability $\rho(P)$ to each projection operator P amounts to assigning a real number between 0 and 1:

$$0 \leq \rho(P) \leq 1 \qquad \text{for each } P. \tag{1}$$

If P and Q are mutually orthogonal projection operators, then $P+Q$ is also a projection operator. Mutually orthogonal projection operators represent mutually exclusive outcomes, and their sum $P+Q$ represents the outcome "either P or Q occurs." For this interpretaiton to be consistent with the assignments (1), we should require

$$\rho(P+Q) = \rho(P) + \rho(Q) \qquad \text{whenever } PQ = 0, \tag{2}$$

$$\rho(\text{identity operator}) = 1. \tag{3}$$

Theorems have been proven about the mathematical implications of these conditions. The most famous of these is **Gleason's theorem**.[6] Gleason's theorem applies to the simplest type of von Neumann algebra, namely type I. More general theorems proven by Christensen (1982) and Yeadon (1983-1984) extend the result to von Neumann algebras of all types: I, II, and III. An even more general theorem proven in Bunce and Wright (1992) assumes only the condition (2), without assuming (1) or (3).

According to these theorems, if an assignment of real numbers to projection operators satisfies conditions (1)-(3), then the assignment can be extended to a **normalized positive linear functional**, which is a special way of assigning complex numbers to *all* of the operators in the von Neumann algebra, not just to the projection operators. This article introduces the concept of a normalized positive linear functional, more concisely called a **state**. This is the mathematical structure that quantum theory uses to assign probabilities to the various possible outcomes of a measurement, based on whatever we already know about the current status of the physical system.

[6] Gleason's theorem is reveiwed in Hamhalter (2003). A relatively concise and accessible proof is given in Cooke *et al* (1985).

2 States

As in article 74088, let A^* denote the adjoint of an operator A. A **normalized positive linear functional** ρ is something that takes any operator A as input and returns a complex number $\rho(A)$ as output, subject to these conditions:

- ρ is **normalized**, which means $\rho(1) = 1$.
- ρ is **positive**, which means $\rho(A)$ is a positive real number whenever A is a positive operator. (An operator A is called positive if $A = B^*B$ for some operator B).
- ρ is **linear**, which means $\rho(A + B) = \rho(A) + \rho(B)$ and $\rho(zA) = z\rho(A)$ for all operators A, B and all complex numbers z.

"Normalized positive linear functional" is a long name. Even experts think the name is too long, so they often just call ρ a **state**.[7]

The conditions listed above imply:

- $\rho(A^*)$ is the complex conjugate of $\rho(A)$.

To prove this, consider the operator $B \equiv z + A$ where z is a complex number (times the identity operator). The quantity $\rho(B^*B)$ must be a real number because ρ is positive, and it must satisfy

$$\rho(B^*B) = z^*z + \rho(A^*A) + \big(z^*\rho(A) + z\rho(A^*)\big)$$

because ρ is normalized and linear. Together, these imply that the term in large parentheses must be a real number, and this must be true for all complex numbers z, so $\rho(A^*)$ must be the complex conjugate of $\rho(A)$.

[7]In physics, the word *state* is often used to refer to the actual status of the physical system, independently of how much we happen to know about it, but no such connotation is intended here. In practice, the (mathematical) state ρ is used to represent whatever we happen to know about the status of the physical system, without presuming that the description is "complete." We still call it a *state*.

3 Constructing states

If $|\psi\rangle$ is a nonzero element of the Hilbert space $\mathcal{H}$, then the functional[8]

$$\rho(\cdots) = \frac{\langle\psi|\cdots|\psi\rangle}{\langle\psi|\psi\rangle} \tag{4}$$

is a state, and then $|\psi\rangle$ is called the **state-vector**. Notice that multiplying $|\psi\rangle$ by a nonzero complex number z has no effect on the state (4), because the factors of z cancel between the numerator and denominator. In other words, for any nonzero complex number z, the state-vectors $|\psi\rangle$ and $z|\psi\rangle$ represent the same state.

More generally, if $|a_1\rangle$, $|a_2\rangle$, ... is any list of nonzero vectors, then the functional defined by

$$\rho(\cdots) = \frac{\sum_n \langle a_n|\cdots|a_n\rangle}{\sum_n \langle a_n|a_n\rangle} \tag{5}$$

is a state. Similarly, if $\rho_1, \rho_2, ..., \rho_N$ are states and $\lambda_1, \lambda_2, ..., \lambda_N$ are positive real numbers that add up to 1, then

$$\rho(\cdots) \equiv \sum_n \lambda_n\, \rho_n(\cdots)$$

is another state. The state (5) has this form with $\rho_n(\cdots) = \langle a_n|\cdots|a_n\rangle/\langle a_n|a_n\rangle$ and $\lambda_n = \langle a_n|a_n\rangle/\sum_k \langle a_k|a_k\rangle$.

If ρ is a state and A is any operator for which $\rho(A^*A) \neq 0$, then the functional defined by

$$\boxed{\rho(\cdots|A) \equiv \frac{\rho(A^*\cdots A)}{\rho(A^*A)}} \tag{6}$$

is another state. The notation $\rho(\cdots|A)$ isn't standard, but it's useful. It is deliberately similar to the standard notation for a conditional probability, which is related to how the construct (6) is used in quantum theory.

[8] Page 7 in Fewster and Rejzner (2019) calls this a **vector state**.

4 The Cauchy-Schwarz inequality

Any state ρ satisfies

$$\boxed{|\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B)} \tag{7}$$

for all operators A and B. This is called the **Cauchy-Schwarz inequality**. It is related to another inequality with the same name that was introduced in article 90771.

To derive (7), use the fact that ρ is linear to get

$$\rho\Big((A - zB)^*(A - zB)\Big) = \rho(A^*A) - z\,\rho(A^*B) - z^*\,\rho(B^*A) + |z|^2\rho(B^*B)$$

and use the fact that ρ is positive to get

$$\rho\Big((A - zB)^*(A - zB)\Big) \geq 0.$$

Both of these relationships hold for any pair of operators A, B and any complex number z. Set $z = \rho(B^*A)/\rho(B^*B)$ to complete the derivation.

5 States and projection operators

In quantum theory, projection operators are used to represent the possible outcomes of a measurement, and a state is used to assign probabilities to those outcomes based on whatever we already know about the current status of the physical system. If P is a projection operator and ρ is a state, then $\rho(P)$ is the probability that quantum theory assigns to this possible outcome.

For this to make sense, the quantity $\rho(P)$ must satisfy

$$\boxed{0 \leq \rho(P) \leq 1.} \tag{8}$$

To see that this condition is satisfied, use the fact that ρ is linear and normalized to get $\rho(P) + \rho(1 - P) = 1$, and then use the identities $P = P^*P$ and $1 - P = (1 - P)^*(1 - P)$ together with the fact that ρ is positive to get $\rho(P) \geq 0$ and $\rho(1 - P) \geq 0$. Altogether, this implies (8), because two non-negative numbers cannot add up to 1 unless both of them are ≤ 1.

The notation introduced in equation (6) is deliberately similar to the standard notation for a conditional probability. The resemblance is strengthened by this result:[9]

$$\text{The condition } \rho(P) = 1 \text{ implies } \rho(\cdots) = \rho(\cdots|P). \tag{9}$$

Proof: The complementary projector $Q \equiv 1 - P$ satisfies $\rho(Q) = 0$. Linearity implies $\rho(PAP) = \rho(A) - \rho(QA) - \rho(AQ) + \rho(QAQ)$. The Cauchy-Schwarz inequality combined with $\rho(Q^*Q) = \rho(Q) = 0$ gives $\rho(Q\cdots) = \rho(\cdots Q) = 0$. Altogether, this gives $\rho(PAP) = \rho(A)$, which immediately implies (9).

[9] The real reason for the notation comes from the postulates of quantum theory, which this article only vaguely foreshadows.

6 References

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7 References in this series

Article **74088** (https://cphysics.org/article/74088):
"Linear Operators on a Hilbert Space" (version 2022-10-23)

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