The Action for Models with Non-Product Gauge Groups and Multiple Coupling Constants

Randy S

Abstract This article explains how to construct an action for a model of quantum gauge fields with gauged group $(G_1 \times \cdots \times G_K)/\Gamma$, where each G_k is a compact Lie group – either $U(1)$ or a simple 1-connected Lie group – and Γ is a discrete subgroup of the center of $G_1 \times \cdots \times G_K$. The action may involve several independent coupling constants, one for each factor G_k and one for each pairing of $U(1)$ factors (if the numerator has more than one $U(1)$ factor). The factor Γ affects the set of possible interaction terms, but it doesn't affect the Lie algebra.

One of the motives for this article is the fact that the gauged group of the Standard Model(s) of particle physics has the form $(SU(3) \times SU(2) \times U(1))/\Gamma$. More than one choice for Γ is consistent with everything that is currently known about the real world, but the most likely choice is a group with six elements for which the resulting gauged group is not a product.

Contents

1 Introduction

The action for a model with gauge fields is a sum of two types of term: terms involving only the gauge fields, and terms that also involve other fields (including the gauge fields). These will be called **Yang-Mills** terms and **interaction** terms, respectively.[1] When spacetime is treated as a lattice so that the quantum model can be constructed nonperturbatively, the standard form for the Yang-Mills terms is called the **Wilson action**.[2]

Any compact connected Lie group has the form[3,4]

$$G = \frac{G_1 \times G_2 \times \cdots \times G_K}{\Gamma} \tag{1}$$

where Γ is a discrete subgroup of the center[5] of $G_1 \times G_2 \times \cdots \times G_K$ and each factor G_k is either $U(1)$ or a simple 1-connected Lie group.[6] Article 89053 constructed an example of a model with gauge fields for a generic compact Lie group G. Before the continuum limit is taken, the action used in that construction has one continuously adjustable dimensionless[7] parameter, namely the overall coefficient of the action, regardless of the number K of factors in (1). This article introduces a more general family of models that have the same gauged group (1) but whose action uses at least[8] K independent continuously adjustable dimensionless parameters, one for each factor G_k.

[1] This differs from the usual meaning of *interaction term* in perturbation theory, where it refers to any term of higher-than-quadratic order in the fields.

[2] Article 89053

[3] Article 92035

[4] Section 5 will review the concept of a quotient group A/B, where B is a normal subgroup of A.

[5] Section 5 will define *center*.

[6] Section 4 will review the definition of *1-connected*, and section 5 will review the definition of *simple*.

[7] Taking a continuum limit converts this parameter to a scale, a phenomenon called **dimensional transmutation** (article 07611).

[8] The number of independent parameters can be higher if two or more of the factors G_k are isomorphic to $U(1)$.

2 Motivation: the Standard Model(s)

The gauged group of the Standard Model(s) of particle physics is[9]

$$\frac{SU(3)\times SU(2)\times U(1)}{\Gamma}$$

where the $SU(3)$ gauge field mediates the strong interaction between quarks, and different mixtures of the $SU(2)$ and $U(1)$ gauge fields mediate the electromagnetic and weak nuclear interactions.[10] A particular six-element subgroup of the center of the numerator is the most conservative choice[11,12] for the denominator Γ because this choice limits the set of possible interaction terms as much as possible without compromising the model's consistency with experiment.[13] Other experimentally allowed choices for Γ may also be considered, though care must be taken to check for *global anomalies* that would make the model mathematically inconsistent in subtle ways that small-coupling expansions do not notice.[14]

One of the goals of a Grand Unified Theory (GUT) is to derive the values of all of the gauge-field coupling constants from a single coupling constant for a larger gauged group. The challenge is then is to find the right combination of interaction terms that makes the GUT consistent with experiment at accessible energies. The purpose of this article is to help explain why a model defined on a spacetime lattice can have multiple *running couplings*,[15] including one for each of the factors G_k in (1), even if the action has only a single Wilson term for the whole gauged group (1). The basic message is that even without considering running couplings, we can define a model whose action has a separate Wilson term for each of the factors G_k, with independent coefficients, without changing the fact that the full gauged group is still (1) and without changing the model's properties in the continuum limit.

[9] Davighi *et al* (2020), equation (4.2); Lohitsiri (2020), equation (3.30); Tong (2017)

[10] These mixtures are selected by interactions between the gauge fields and the Higgs field(s).

[11] Harlow and Ooguri (2021), footnotes 28 and 58 in sections 2.4 and 3.4 (in the preprint version)

[12] This choice is also motivated by Grand Unified Theories (Tong (2018), text below equation (2.82)).

[13] Making Γ larger makes the set of allowed interaction terms smaller.

[14] Davighi *et al* (2020), section 4.6; Lohitsiri (2020), section 3.4

[15] Article 07611

3 Preview and perspective

The Wilson action uses a faithful matrix representation of the gauged group G. Interaction terms and observables may use representations of the gauged group G that are not necessarily faithful.

The continuum limit of the Wilson action "forgets" about the denominator Γ in (1).[16] More precisely, sections 13-15 will show that under mild conditions on the factors G_k,[17] the continuum limit of the Wilson action becomes a sum of Yang-Mills terms, one for each factor G_k in the numerator of (1):

$$S_W(G,q) \to \sum_k S_{YM}(G_k,\, q_k) \qquad\qquad \text{(continuum limit)}, \tag{2}$$

where $S_W(G,q)$ is the Wilson action for a gauge field when the gauged group is G and the coupling constant is q, and $S_{YM}(G,q)$ denotes the continuum limit of $S_W(G,q)$. The denominator Γ of (1) still affects the set of allowed interaction terms,[18,19] but it doesn't affect the continuum limit of the Yang-Mills terms except through relationship between the constants q_k and q.

This relationship between the constants q_k and q constrains the properties of the model's continuum limit. The "constants" q_k may flow at different rates under

[16] Beware that the continuum limit of a lattice model is not necessarily fully determined by the continuum limit of its action. The denominator Γ in (1) affects the topology of G, so it can affect the set of possible principal G-bundles when the base space (spacetime) is topologically nontrivial. Roughly, making Γ larger make the topology of G more complicated and so can allow a larger variety of principle G-bundles. The path integral in continuous spacetime – if we knew how to define it – would include a sum over bundles (Aharony *et al* (2013), section 1.1, property 3), so Γ could affect the model's properties on topologically nontrivial spacetimes. Teper (2018) didn't find any differences between pure Yang-Mills models (in which the gauge field is the only field) whose gauged groups differ only in the denominator Γ, but that analysis only considered a limited class of observables.

[17] One of the conditions is that no more than one $U(1)$ factor is present. If two or more $U(1)$ factors are present, then some choices of the denominator Γ can lead to **kinetic mixing terms** (section 17).

[18] Interaction terms are not shown in this article.

[19] Each interaction term uses a representation of G/Γ. The set of representations of G/Γ is a subset of the set of representations of G (section 6). Making Γ bigger makes the subset smaller.

the renormalization group,[20,21] making the relationship between them less apparent at resolutions much coarser than the lattice scale, but the constraint is still there.

Section 16 will describe a lattice action that treats the coupling constants q_k as independent parameters instead of using (2) to derive their values from the value of a single governing constant q.[22] The gauged group is still (1), even though we now have K independent coupling constants instead of only one. Here's an outline of the rest of the article:

- Sections 4-11 review some definitions and general results about Lie groups, Lie algebras, and their representations.
- Section 12 reviews the Wilson action, and sections 13-15 derive the result (2).
- Section 16 explains how to modify the lattice action so that it has the same continuum limit (2) but allows the coupling constants q_k to be treated as independent parameters.
- Section 17 addresses an exception to (2), namely when the numerator of (1) has two or more $U(1)$ factors.
- Sections 18-28 use several examples of how to construct faithful matrix representations of a group of the form (1) starting with faithful representations of the factors G_k.

[20] Article 10142 introduces the idea of renormalization group flows in the simpler context of scalar fields, and article 07611 quantifies the flow in the context of a Yang-Mills model with a single coupling constant.

[21] The rates depend both on the gauged group and on the interaction terms.

[22] Section 1 in Baez and Huerta (2010) asserts "When G is the product of simple factors, there is one coupling constant for each factor of G." This article gives more insight about that assertion.

4 Notation and conventions

- A group is called **trivial** if it has only one element (the identity element).
- A topological space (like a Lie group) is called **1-connected** if it is both connected[23] and simply-connected.[24,25]
- $\simeq$ denotes isomorphism of Lie groups or Lie algebras.[26]
- $Z(G)$ is the center of the group G.[26]
- $\mathbb{Z}_n$ is the cyclic group with n elements.
- ρ is a matrix representation of a group G, and $\rho(g)$ is the matrix representing the element $g \in G$.[27]
- ρ^* is the representation **dual** to ρ.[28]
- 1 is the identity matrix, used as the identity element of a matrix group.
- M^T is the transpose of a matrix M.
- $M^\dagger$ is the hermitian conjugate of a matrix M (the complex conjugate of M^T).
- $\langle M\rangle$ denotes the trace of a matrix M.
- $L(G)$ is the Lie algebra corresponding to a Lie group G.[29,30]

---

[23] *Connected* means that any pair of points may be continuously morphed to a single point.

[24] *Simply-connected* means that any closed loop may be continuously morphed to a single point.

[25] Article 61813

[26] Section 5

[27] Section 6

[28] Section 18

[29] Section 9

[30] This notation allows referring to the Lie algebra of a group that hasn't been given a single-letter name of its own. (Contrast this with the common convention that writes $\mathfrak{g}$ for the Lie algebra of G.) Example: the Lie algebra of $(G_1 \times G_2)/\Gamma$ is $L((G_1 \times G_2)/\Gamma)$.

5 Compact Lie groups

Article 92035 reviews the definition of **compact Lie group**. Here's a quick review of other concepts that are useful in this context:

- The **center** of a group G, denoted $Z(G)$, consists of all elements of G that commute with every element of G.

- If G and H are groups, then a **homomorphism** ρ from G to H is a map from G to H that satisfies $\rho(a)\rho(b) = \rho(ab)$ for all $a, b \in G$.[31] The **kernel** of ρ is the set of elements of G for which $\rho(g)$ is the identity element of H.

- Two groups G and H are **isomorphic** if homomorphisms $G \to H$ and $H \to G$ exist for which the compositions $G \to H \to G$ and $H \to G \to H$ are both the identity map.[31]

- A subgroup of G is called a **normal** subgroup if it is the kernel of some homomorphism.[31] A Lie group G is called **simple** if it doesn't have any nontrivial connected normal subgroups other than G itself.[32]

- A Lie group Γ is called **discrete** if each element of Γ has a neighborhood that doesn't contain any other elements. If G is a connected matrix Lie group and Γ is a discrete normal subgroup of G, then Γ is in the center of G.[33]

- If G is a group and Γ is a normal subgroup, then the **quotient group** G/Γ is "G modulo Γ." Article 29682 gives the precise definition.

- A Lie group G is called **semisimple** if it has the form (1) in which all the factors G_k are simple and 1-connected.[32] If G is semisimple, then the center of $G/Z(G)$ is trivial.[34]

[31] Article 29682
[32] Article 92035
[33] Hall (2015), chapter 1, exercise 11
[34] Knapp (2023), proposition 6.30

6 Representations of compact Lie groups

In this article, a **representation** of a group G means a finite-dimensional matrix representation over $\mathbb{C}$ – a homomorphism from G to a matrix group with complex numbers as matrix elements. To avoid misunderstanding, the qualifiers *finite-dimensional* or *matrix* are still included when quoting some theorems, even though those qualifiers are implied throughout this article.

- Two representations ρ and ρ' of G are called **equivalent** if a matrix M exists for which $\rho'(g) = M^{-1}\rho(g)M$ for all $g \in G$.[35,36]
- The **kernel** of a representation ρ of G consists of all elements $g \in G$ for which $\rho(g) = 1$.
- A representation ρ of G is called **faithful** if the condition $g \neq g'$ implies $\rho(g) \neq \rho(g')$.[37]
- Every compact Lie group G has a faithful finite-dimensional representation.[38]
- A representation ρ of G is called **unitary** if $\rho(g^{-1}) = (\rho(g))^\dagger$ for every $g \in G$.[39]
- Every finite-dimensional representation (over $\mathbb{C}$) of a compact Lie group is equivalent to a unitary representation.[40] In this article, all representations are assumed to be unitary.
- If G is a group and Γ is a normal subgroup, then any representation ρ of G/Γ (homomorphism from G/Γ to a matrix group) gives a representation of G by composing it with the homomorphism $G \to G/\Gamma$, so the set of representations of G/Γ is a subset of the set of representations of G.

[35]Kirillov and Kirillov (2005), section 4.1

[36]The text below definition 4.3 in Hall (2015) calls equivalent representations *isomorphic*. I'm using the name *equivalent* to help avoid misunderstanding, Two Lie algebras are not necessarily isomorphic to each other even if they have some representations that are equivalent to each other (section 9).

[37]Hall (2015), section 4.1

[38]Knapp (2023), corollary 4.22

[39]Hall (2015), definition 4.7

[40]Kirillov and Kirillov (2005), theorem 11

7 Direct product, direct sum, and tensor product

The **direct product** of two groups A and B is denoted $A \times B$. Each element of $A \times B$ is a pair (a, b) with $a \in A$ and $b \in B$, and the group operation is defined by $(a, b)(a', b') = (aa', bb')$. This definition does not rely on the concept of a matrix representation.

The *direct sum* and *tensor product* are two different ways of combining representations. Both may be defined by how they combine the vector spaces on which the representations act.[41] Use this notation:

- A is a group, and ρ_A is a representation of A using matrixes[42] that act on[43] an N_A-dimensional vector space V_A over $\mathbb{C}$.

- B is another group, and ρ_B is a representation of B using matrixes that act on an N_B-dimensional vector space V_B over $\mathbb{C}$.

The **direct sum** of the vector spaces, denoted $V_A \oplus V_B$, is defined by thinking of V_A and V_B as linearly independent subspaces of an $(N_A + N_B)$-dimensional vector space. The **direct sum** $\rho \equiv \rho_A \oplus \rho_B$ of the representations ρ_A and ρ_B is the representation of $A \times B$ defined by[44]

$$\rho(a, b)(V_A \oplus V_B) = \big(\rho_A(a)V_A\big) \oplus \big(\rho_B(b)V_B\big) \tag{3}$$

for all $(a, b) \in A \times B$. We can use a basis in which the first N_A components of a vector in $V_A \oplus V_B$ are the components of a vector in V_A, and the last N_B components of a vector in $V_A \oplus V_B$ are the components of a vector in V_B. In this basis, $\rho(a, b)$

[41] Article 28539 reviewed definitions of the direct sum and tensor product of two *abelian groups*. The definitions in that article are consistent with the definitions in this article if the roles of the *abelian groups* in that article are compared to the roles of the *vector spaces* in this article. (A vector space is, among other things, an abelian group.) However, here we are defining the direct sum and tensor product of two *representations* (in terms of the direct sum and tensor product of the vector spaces on which they act), not of the groups they represent.

[42] I'm writing the plural of matrix as *matrixes* instead of *matrices* to remind students that *matrice* is not a word.

[43] Saying that a matrix M **acts on** a vector space V means that it describes a linear transformation of V.

[44] Hall (2015), definition 4.12

is the block-diagonal matrix whose diagonal blocks are $\rho_A(a)$ and $\rho_B(b)$. If ρ_A and ρ_B are faithful representations of A and B, then $\rho_A \oplus \rho_B$ is a faithful representation of $A \times B$.[45]

The **tensor product** of V_A and V_B, denoted $V_A \otimes V_B$, can be defined as the $(N_A N_B)$-dimensional vector space in which the components of a vector $v \in V_A \otimes V_B$ are $v_{(j,k)} = \alpha_j \beta_k$ with $\alpha \in V_A$ and $\beta \in V_B$. In this description, the pair (j,k) of integers is regarded as a single index for the components of v. This single index ranges over $N_A N_B$ different values, because j and k range over N_A and N_B different values, respectively. In this basis, the **tensor product** of a matrix M_A acting on V_A with a matrix M_B acting on V_B is the matrix $M = M_A \otimes M_B$ whose action on $V_A \otimes V_B$ can be defined componentwise by[46]

$$\big[M(V_A \otimes V_B)\big]_{(j,k)} = \big[M_A V_A\big]_j \big[M_B V_B\big]_k. \tag{4}$$

The **tensor product** $\rho \equiv \rho_A \otimes \rho_B$ of the representations ρ_A and ρ_B is the representation of $A \times B$ defined by[47]

$$\rho(a,b) = \rho_A(a) \otimes \rho_B(b) \tag{5}$$

for all $(a,b) \in A \times B$, where the right-hand side is a matrix defined by (4). Even if ρ_A and ρ_B are faithful representations of A and B, $\rho_A \otimes \rho_B$ might not be a faithful representation of $A \times B$, because $(zM_A) \otimes M_B = M_A \otimes (zM_B)$ for all complex numbers z.[48]

We can also use the direct sum or tensor product to combine two representations of a single group. If ρ_A and ρ_B are both representations of G, then the representations $\rho_A \oplus \rho_B$ and $\rho_A \otimes \rho_B$ of G are defined by $\rho(g) \equiv \rho_A(g) \oplus \rho_B(g)$ and $\rho(g) \equiv \rho_A(g) \otimes \rho_B(g)$,[49] respectively.

[45] This should be clear from the block-diagonal description.

[46] A basis-independent definition is given in Hall (2015), definition 4.13

[47] Hall (2015), definition 4.17

[48] This property is sometimes indicated by writing $\otimes_{\mathbb{C}}$, where the subscript specifies the type of number that may be passed from one side of the tensor product to the other. This allows distinguishing between $\otimes_{\mathbb{C}}$, $\otimes_{\mathbb{R}}$, and $\otimes_{\mathbb{Z}}$, which can be useful in some contexts. In this article, $\otimes$ always means $\otimes_{\mathbb{C}}$.

[49] Hall (2015), definition 4.20

8 General form of a finite-dimensional representation

Let ρ be a representation of G acting on a vector space V. The representation ρ is called **irreducible** if V does not have any subspace (other than the zero-dimensional subspace and V itself) that is self-contained under the action of ρ.[50] A representation is called **completely reducible** if it may be written as a direct sum of a finite number of irreducible representations.[51] If G is a compact matrix Lie group, then every finite-dimensional representation of G is completely reducible.[52]

If ρ is an irreducible complex representation of a matrix Lie group G and g is in the center of G, then $\rho(g)$ is proportional to the identity matrix.[53]

If ρ_1 and ρ_2 are finite-dimensional irreducible representations of G_1 and G_2, then $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \times G_2$, and every finite-dimensional irreducible representation of $G_1 \times G_2$ has this form.[54] This generalizes in the obvious way to direct products with any finite number of factors.

If G is a direct product of compact matrix Lie groups G_k,

$$G = G_1 \times \cdots \times G_K, \tag{6}$$

then G is also a compact matrix Lie group, so all finite-dimensional representations of G are completely reducible. Every irreducible representation of (6) is a tensor product of irreducible representations of its factors G_k, so every finite-dimensional representation of (6) may be expressed as a direct sum of tensor products of representations of its factors G_k. Every representation of G/Γ is also a representation of G,[55] so every representation of G/Γ may also be expressed as a direct sum of tensor products of representations of the factors G_k in the numerator. Sections 19-28 will describe several examples.

[50] Hall (2015), definition 4.2

[51] Hall (2015), definition 4.23

[52] Hall (2015), Theorem 4.28; Knapp (2023), theorem 9.4

[53] Hall (2015), Corollary 4.30

[54] Sepanski (2007), theorem 3.9 (for compact Lie groups); Morel (2019), theorem II.2.1 (the proof is given only for finite groups, but the theorem is stated without that restriction)

[55] Section 6

9 The Lie algebra of a Lie group

Every Lie group has a corresponding Lie algebra. The **Lie algebra** $L(G)$ of a matrix group G can be defined as the set of all matrixes X for which e^{sX} is in G for all real numbers s,[56] and then the **Lie bracket** of $X, Y \in L(G)$ can be defined as the commutator $[X, Y]$.[57]

If G is connected, then $L(G)$ **generates** G,[58] but an abstract Lie algebra doesn't generate a Lie group. In other words: every connected Lie group G is generated by its Lie algebra $L(G)$,[59] but two connected Lie groups may be non-isomorphic even if their Lie algebras are isomorphic. In particular, if Γ is a discrete subgroup of the center of G, then[60]

$$L(G/\Gamma) \simeq L(G), \tag{7}$$

even though G/Γ is typically not isomorphic to G.

If G is a matrix Lie group, then every representation of G gives a representation of $L(G)$,[61] but different faithful matrix representations of $L(G)$ can generate groups that are not isomorphic to each other. Example: let ρ_3 and ρ_2 be faithful representations of the matrix groups $SO(3)$ and $SU(2)$, respectively. Then ρ_3 and ρ_2 give faithful representations of $L(SO(3))$ and $L(SU(2))$, respectively. Equation (7) implies $L(SO(3)) \simeq L(SU(2))$ because $SO(3) \simeq SU(2)/\mathbb{Z}_2$, so ρ_3 and ρ_2 are two different faithful representations of the same abstract Lie algebra, even though the groups they generate ($SO(3)$ and $SU(2)$) are not isomorphic to each other.

[56] Hall (2015), definition 3.18 or 5.18

[57] These aren't the general definitions, but they're sufficient for *matrix* Lie groups/algebras.

[58] This means that every element of the group G may be written as a product of elements of the form e^X with $X \in L(G)$ (Hall (2015), corollary 3.47).

[59] Fulton and Harris (1991), section 8.3, page 116

[60] Fulton and Harris (1991), section 8.3, page 119

[61] Partial converse: if G is a connected and simply-connected matrix Lie group, then every representation of $L(G)$ comes from a representation of G (Hall (2015), theorem 5.6, previewed at the beginning of section 4.7).

10 Faithful representations of a Lie algebra

Consider a unitary matrix representation of a Lie group G. If X is a matrix in the corresponding representation of $L(G)$, then X is antihermitian: $X^\dagger = -X$.[62]

Now let $T_1, T_2, ...$ be linearly independent elements of $L(G)$ such that every element of $L(G)$ is a linear combination of the T_as with real numbers as coefficients. If ρ is a faithful unitary matrix representation of $L(G)$, then the matrixes $\rho(T_1)$, $\rho(T_2)$, ... are also linearly independent over $\mathbb{R}$ (the field of real numbers), and we can choose the basis $T_1, T_2, ...$ so that

$$\langle \rho(T_a)\rho(T_b)\rangle = \begin{cases} -\nu & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for some positive real constant ν, where $\langle\cdots\rangle$ denotes the trace of a matrix.[63]

If G is a compact semisimple Lie group, then every element $X \in L(G)$ may be written as $X = [X_1, X_2]$ with $X_1, X_2 \in L(G)$.[64,65] This implies that in any faithful matrix representation of $L(G)$, every element of $L(G)$ is represented by a traceless matrix, because the trace of a commutator is zero. This applies to all the factors G_k in the numerator of (1) with the exception of the $U(1)$ factors.

[62] This follows from the fact that if s is a real number, then e^{sX} is a matrix representing an element of G, and this matrix must be unitary.

[63] Choosing the basis so that the trace is zero for $a \neq b$ is possible because the matrix M with components $M_{ab} \equiv \langle \rho(T_a)\rho(T_b)\rangle$ is symmetric (because the trace is cyclic) and real, so it can be diagonalized by an orthogonal transformation. The constant ν must be positive because $\langle \rho(T_a)^\dagger \rho(T_a)\rangle$ is positive and $\rho(T_a)^\dagger = -\rho(T_a)$.

[64] Malkoun and Nahlus (2017), theorem 1.1

[65] This is the Lie algebra counterpart of **Gotô's theorem** for compact connected semisimple lie groups (Gotô (1949); Kramer (2023), text above theorem 1).

11 The adjoint representation

Every matrix Lie group has a representation called the **adjoint representation**. To define it, first define an action of G on its Lie algebra $L(G)$ by

$$\rho(g) : X \mapsto gXg^{-1} \qquad \text{for all } X \in L(G) \text{ and } g \in G. \tag{9}$$

If we think of $L(G)$ as a vector space on which $\rho(g)$ acts, then ρ is a representation of G. This is the adjoint representation.[66]

To make this more explicit, let $X_1, X_2, ...$ be a basis for $L(G)$ matrixes satisfying $\langle X_j X_k \rangle = -\delta_{jk}$, like in section 10. Every $X \in L(G)$ is a linear combination of these basis elements with coefficients in $\mathbb{R}$. The definition (9) says that $\rho(g)$ sends X_j to gX_jg^{-1}. This must be a linear combination of these basis elements, so

$$gX_jg^{-1} = \sum_j M_{jk}(g) X_k \tag{10}$$

with coefficients $M_{jk}(g) \in \mathbb{R}$. We can think of $L(G)$ as a vector space with basis vectors X_j, and we can represent X_j as the vector whose kth component is δ_{jk}. Then the effect of (9) is the same as multiplying these vectors by the transpose $M^T(g)$ of the matrix $M(g)$ with components $M_{jk}(g)$, so $\rho(g) = M^T(g)$.[67]

The adjoint representation is a faithful representation of $G/Z(G)$.[68] Combine this with (7) to deduce that the adjoint representation of G gives a faithful representation of $L(G)$.

[66] Hall (2015), definitions 3.32 and 4.9

[67] If g is represented by a unitary matrix, then $\rho(g)$ is also unitary. Proof: $\delta_{jk} = -\langle X_j X_k \rangle = -\big\langle (gX_jg^\dagger)(gX_kg^\dagger) \big\rangle = -\sum_{j'k'} \langle M_{jj'} X_{j'} M_{kk'} X_{k'} \rangle = \sum_\ell M_{j\ell} M_{k\ell}$, so M is an orthogonal matrix. Its components are real, so M is unitary.

[68] Article 91563

12 Overview of the derivation of (2)

Let ρ be a faithful matrix representation of a compact Lie group, and let $N(\rho)$ denote the trace of the identity matrix in this representation. The Wilson action is a function of a set of G-valued variables, with one variable matrix $u(\rho, \Box)$ for each plaquette $\Box$ in the lattice.[69] The Wilson action is

$$S_W(\rho, q) = \frac{\beta}{2} \sum_{\Box} \left(1 - \frac{W(\rho, \Box)}{N(\rho)}\right) \tag{11}$$

with $\beta \propto 1/q^2$ and

$$W(\rho, \Box) = \big\langle u(\rho, \Box) \big\rangle.$$

The goal is to express the continuum limit of this action in terms of Yang-Mills actions for the individual factors G_k in (1).

Every representation of (1) may also be expressed as a direct sum of tensor products of representations of the factors G_k in the numerator.[70] One of the steps in section 15 uses the fact that for each factor G_k, the representations that occur in that decomposition are all either trivial or faithful representations of the Lie algebra of G_k. This is true because each non-$U(1)$ factor G_k is assumed to be a simple Lie group, which means it doesn't have any connected normal subgroups. The kernel of a representation must be a normal subgroup, so every nontrivial representation of G_k must be a faithful representation of the Lie algebra of G_k.

Sections 13 and 14 decompose the Wilson action for a direct sum and a direct product, respectively, and then section 15 takes the continuum limit. The result is (2), plus possible cross-terms that only involve $U(1)$ factors.

[69] These plaquette variables are not all independent of each other. They are special combinations of link variables, and the link variables are all independent of each other (article 89053).

[70] Section 8

13 Wilson action for a direct sum

Now suppose $\rho = \rho^{(1)} \oplus \rho^{(2)} \oplus \cdots$, with a finite number of terms, and think of each summand as a block in a basis where ρ is block-diagonal. This gives $N(\rho) = \sum_j N\left(\rho^{(j)}\right)$ and

$$u(\rho,\square) = u\left(\rho^{(1)},\square\right) \oplus u\left(\rho^{(2)},\square\right) \oplus \cdots \qquad \Rightarrow \qquad W(\rho,\square) = \sum_j W\left(\rho^{(j)},\square\right),$$

where the traces in $N\left(\rho^{(j)}\right)$ and $W\left(\rho^{(j)},\square\right)$ are over the kth block of the direct sum. Use these to get

$$\begin{aligned} S_W(\rho,\beta) &= \frac{\beta}{2}\sum_{\square}\frac{N(\rho)-W(\rho,\square)}{N(\rho)} \\ &= \frac{\beta}{2}\sum_{\square}\sum_{j}\frac{N\left(\rho^{(j)}\right)-W\left(\rho^{(j)},\square\right)}{N(\rho)} \\ &= \sum_j \frac{\beta_j}{2}\sum_{\square}\left(1-\frac{W\left(\rho^{(j)},\square\right)}{N\left(\rho^{(j)}\right)}\right) \end{aligned} \tag{12}$$

with $\beta_j \equiv \beta N\left(\rho^{(j)}\right)/N(\rho)$.

14 Wilson action for a tensor product

Let ρ denote one of the terms $\rho^{(j)}$ in the direct sum in section 13, omitting the superscript to reduce clutter. Suppose that ρ has the form

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \tag{13}$$

with a finite number of factors, where each ρ_m is a representation of one of the factors G_k in the numerator of (1). The correspondence between factors ρ_m in (13) and factors G_k in (1) does not need to be (and typically won't be) one-to-one. The trace of a tensor product is the product of the traces of the factors,[71] so

$$W(\rho, \square) = \prod_m W(\rho_m, \square) \qquad\qquad N(\rho) = \prod_m N(\rho_m). \tag{14}$$

[71] Hall (2015), lemma 12.17

15 Continuum limit of the action

This section derives the continuum limit of the Wilson action for a finite-dimensional faithful unitary representation of the gauged group (1), using the fact that any such representation may be expressed as a direct sum of tensor products of representations of the factors G_k in the numerator of (1).[72] Sections 13-14 already did the first steps in the derivation. This section finishes it.

Consider one of the factors in the first of equations (14). Article 89053 derives the result

$$W(\rho_m, \square) = \left\langle e^{r(\rho_m,\square)} \right\rangle \qquad\qquad r(\rho_m, \square) \equiv \epsilon^2 F(\rho_m, \square) + O(\epsilon^3), \tag{15}$$

where ϵ is the distance between adjacent lattice sites and $F(\rho_m, \square)$ is an element of the Lie algebra in the representation ρ_m. Expand the exponential e^r in powers of r and substitute the resulting expression for $W(\rho_m, \square)$ into (14). This gives

$$\begin{aligned} W(\rho, \square) = c + \sum_m c'_m \left\langle r(\rho_m, \square) + \frac{1}{2} r^2(\rho_m \square) \right\rangle \\ + \sum_{m<m'} c''_{m,m'} \big\langle r(\rho_m, \square) \big\rangle \big\langle r(\rho_{m'}, \square) \big\rangle + O(\epsilon^6) \end{aligned}$$

where the constants $c, c'_m, c''_{m,m'}$ are products of traces of the identity matrix. The constant term c is canceled by the "1" in equation (12). The terms linear in r cancel in the sum over plaquettes in (12) because $r(\rho_m, \square) + r(\rho_m, \square^{\text{rev}}) = 0$ if $\square^{\text{rev}}$ is the plaquette obtained from $\square$ by reversing the orientation.[73] Terms of order ϵ^5 and higher go to zero in the continuum limit of the action.[73] This leaves

$$\begin{aligned} W(\rho, \square) = \frac{\epsilon^4}{2} \sum_m c'_m \big\langle F^2(\rho_m, \square) \big\rangle + \epsilon^4 \sum_{m<m'} c''_{m,m'} \big\langle F(\rho_m, \square) \big\rangle \big\langle F(\rho_{m'}, \square) \big\rangle \\ + \text{(terms that will cancel or go to zero)}. \end{aligned}$$

[72] Section 8

[73] Article 89053

If ρ_m is a nontrivial representation of one of the non-$U(1)$ factors in the numerator of the gauged group (1), then the trace of $F(\rho_m, \square)$ is zero,[74] so the only surviving cross-terms are those involving two $U(1)$ factors. If ρ_m is a trivial representation, then the trace is $N(\rho_m)$. Altogether, after expressing the coefficients c'_m in terms of traces of identity matrixes, this leaves

$$\sum_{\square}\left(1-\frac{W(\rho,\square)}{N(\rho)}\right) = -\frac{\epsilon^4}{2}\left(\sum_{\square}\sum_{m}\frac{\left\langle F^2(\rho_m,\square)\right\rangle}{N(\rho_m)} + (U(1)\text{ cross-terms})\right) + O(\epsilon^5). \tag{16}$$

Use this in (12) to conclude that continuum limit of the Wilson action for a gauged group of the form (1) is a linear combination of Yang-Mills terms, each proportional to the continuum limit of $\sum_\square \left\langle F^2(\rho_m, \square)\right\rangle$ for some representation ρ_m of (1), plus possible cross-terms that only involve $U(1)$ factors.[75]

To finish deriving (2), consider any factor G_k in the numerator of (1). Equation (16) may have different terms corresponding to different representations ρ_m of the same factor G_k. All these terms are proportional to each other because:

- Each depends only on a representation ρ_m of Lie algebra of G_k, regardless of what part of the center of G_k occurs in the denominator of (1).[76]

- Each representation ρ_m is either trivial or faithful as a representation of the Lie algebra of G_k.[77]

- Equation (8) shows that up to an overall constant factor, $\left\langle F_{ab}^2\right\rangle$ is independent of which faithful representation of the Lie algebra is used.[78]

Since they're all proportional to each other, we can combine all the Yang-Mills terms corresponding to any given factor G_k in (1). This gives (2), plus possible cross-terms between different $U(1)$ factors in (1).

[74] Section 9

[75] Section 17

[76] Equation (7) says that the denominator doesn't affect the Lie algebra if G_k is connected.

[77] Section 12

[78] We might as well take it to be the adjoint representation (section 11).

16 Allowing multiple coupling constants

The derivation in the preceding sections produced an action of the form (2) (plus possible $U(1)$ cross-terms) in which all of the coupling constants q_k are proportional to the coupling constant q in the original action 11.[79] If we're not trying to build a GUT, though, then we might want to treat the coupling constants q_k as being independent of each other instead.[80] That raises a question: how can we modify the lattice model so that it also treats them as independent of each other? We can do this by replacing the Wilson action for the full group G with a sum of Wilson actions for quotients that each involve only one of the factors G_k:

$$S_W(G,q) \to \sum\nolimits_k S_W\left(G_k/\Gamma_k,\, q_k\right) + (U(1)\text{ cross-terms}). \tag{17}$$

Each denominator Γ_k is the smallest subgroup of the center of G_k that contains all the elements of G_k that contribute to Γ, the denominator in (1). The denominators Γ_k in (17) ensure that the right-hand side uses only representations that are induced by the original faithful representation used on the left-hand side.[81] The denominators Γ_k don't determine the denominator Γ of the full group (1), but the denominator Γ of the full group still affects the set of allowed interaction terms (which are not being written in this article).

Even if we use the values of q_k that come from the derivation in section 15 and even if cross-terms between different $U(1)$ factors are absent, the quantum models defined by the left- and right-hand sides of (17) can have different properties on a finite lattice,[82] but the fact that the actions have the same continuum limit is consistent with the assumption that the corresponding quantum models become indistinguishable in appropriate continuum limits.[83,84]

---

[79]The Yang-Mills term for G_k is $\propto 1/q_k^2$.

[80]Section 3

[81]This works because if $G' \equiv G_1 \times \cdots \times G_K$ and $\Gamma' \equiv \Gamma_1 \times \cdots \times \Gamma_k$, then $\frac{G_1}{\Gamma_1} \times \cdots \times \frac{G_K}{\Gamma_K} = \frac{G'}{\Gamma'} = \frac{G'/\Gamma}{\Gamma'/\Gamma} = \frac{G}{\Gamma'/\Gamma}$ (article 29682), so the set of representations of $\frac{G'}{\Gamma'}$ is contained in the set of representations of G (section 6).

[82]Creutz and Moriarty (1982)

[83]I'm not aware of any proof or systematic numerical studies of this assumption.

[84]I avoided saying "in *the* continuum limit" because continuum limits could be taken in different ways when the

17 $U(1)$ cross-terms

Section 15 showed that the Wilson action for the full gauged group (1) reduces to a sum of Yang-Mills actions for the individual factors G_k in the numerator of (1) plus possible cross-terms between different $U(1)$ factors in the numerator of (1).[85] The coefficients of Yang-Mills terms for the individual factors G_k and the $U(1)$ cross-terms may all flow at different rates under the renormalization group,[86] depending on what interaction terms (terms describing interactions of other fields with the gauge fields) are present in the action. In particular, even if we start with a lattice model whose action has no such cross-terms, cross-terms between different $U(1)$ factors will typically arise in the low-resolution effective model generated by the renormalization group flow. This is called **kinetic mixing**. Such cross-terms commonly arise in low-energy effective models derived from GUTs or from string theory.[87] If we're not trying to build a GUT, then we can treat the coefficients of all of these terms – including the $U(1)$ cross-terms – as independent parameters, just like in section 16.

---

lattice action has multiple independent parameters. De Cesare *et al* (2021) considers a lattice action involving two Wilson terms using different representations of a single group and explores the possibility of obtaining different continuum limits (in five-dimensional spacetime) depending on how their coefficients are related to each other in the limit. Florio *et al* (2021) lists some additional references. Emel'yanov and Petrovskiĭ (1983) is another example that explores properties of a model whose lattice action involves multiple representations of the gauged group.

[85] All nontrivial representations of $U(1)$ are faithful representations of the Lie algebra, so cross-terms between different representations of the same $U(1)$ factor are proportional to non-cross-terms. Such terms don't need to be considered separately if the coefficients of all non-cross-terms are being treated as independent parameters anyway.

[86] Section 3

[87] Jaeckel and Ringwald (2010); Luo and Xiao (2003); Babu *et al* (1998)

18 Introduction to the examples

Sections 19-26 give examples of faithful representations of various groups of the form (1). This section reviews a few more tools that can be useful for this purpose.

Every irreducible unitary representation of a compact group is finite dimensional.[88] If ρ_1 and ρ_2 are two irreducible representations of G, then $\rho_1 \otimes \rho_2$ is typically not irreducible[89] as a representation of G.[90]

If ρ is a representation of G, then the **dual representation** ρ^* is defined by[91] $\rho^*(g) \equiv \rho(g^{-1})^T$ for all $g \in G$. If the representation is unitary, then the dual representation is clearly the same as the complex conjugate of ρ, so the notation ρ^* may be interpreted either way when the representation is unitary.

In the following examples, each factor in the numerator of (1) is either $SU(\cdot)$ or $U(1)$. By definition, $SU(n)$ is the matrix group in which each matrix M has size $n \times n$ and satisfies $M^{-1} = M^\dagger$ and $\det M = 1$. The center $Z(SU(n))$ of $SU(n)$ consists of multiples of the identity matrix, where the coefficient z ranges over all complex numbers satisfying $z^n = 1$, so $Z(SU(n))$ is isomorphic to $\mathbb{Z}_n$.[92] The center of the quotient group $SU(n)/Z(SU(n))$ is trivial.[93] Article 91563 reviews some useful facts about the irreducible representations of $SU(n)$,[94] including the number of dimensions of each irreducible representation.[95]

[88] Knapp (2023), Corollary 9.5

[89] Hall (2015), text below definition 4.20

[90] The study of how tensor-product representations of G decompose into irreducible representations of G is sometimes called **Clebsch–Gordan** theory (Hall (2015), text below definition 4.20). Chapter 9 in Knapp (2023) treats this as an application of **branching theorems** that address how irreducible representations of a group G decompose into irreducible representations of a subgroup $H \subset G$.

[91] Hall (2015), definition 4.21 (for matrix Lie groups); Fulton and Harris (1991), section 1.1 (for finite groups)

[92] $SU(n)$ has other subgroups that are also isomorphic to $\mathbb{Z}_n$. One such subgroup consists of matrixes of the form $\text{diag}(z, z^*, 1, 1, 1, ...)$ with $z^n = 1$.

[93] Article 92035

[94] Every unitary representation of $SU(n)$ is a direct sum of irreducible representations.

[95] This information can sometimes be used to show that two quotients G/Γ are not isomorphic to each other even if their denominators Γ are isomorphic to each other, like in section 21.

19 Example: $SU(n)$/center

In this section, Z will be used as an abbreviation for the center of $SU(n)$, which is isomorphic to $\mathbb{Z}_n$. Some representations of $SU(n)$ are also representations of $SU(n)/Z$, and some of those are faithful representations of $SU(n)/Z$. This section explains how to construct one of those faithful representations of $SU(n)/Z$.

Let ρ be the defining representation of $SU(n)$ – the one that was used to define $SU(n)$ in section 18. This a faithful and irreducible representation of $SU(n)$. Let ρ^* denote the dual representation. The tensor product $\rho \otimes \rho^*$ is also a representation of $SU(n)$,[96] but it is not faithful (or irreducible). It is not faithful because in the representation ρ, elements of the center of $SU(n)$ are proportional to the identity matrix,[97] and the proportionality factor (a complex number satisfying $z^n = 1$) can be passed through the tensor product from one side to the other:

$$(z\rho)^* \otimes (z\rho) = (z^*\rho^*) \otimes (z\rho) = \rho^* \otimes (z^* z\rho) = \rho^* \otimes \rho$$

for all complex numbers z satisfying $z^n = 1$ (which implies $z^* z = 1$).

The same identity implies that $\rho^* \otimes \rho$ is also a representation of $SU(n)/Z$. It's not irreducible,[98] but it is a faithful representation of $SU(n)/Z$ because elements of the center of $SU(n)$ are the only factors that can be passed from one side of the tensor product to the other.[99]

[96] In this representation, the matrix that represents $g \in SU(n)$ is $\rho(g) \otimes \rho^*(g)$.

[97] This is true for any irreducible representation of a matrix Lie group (section 8).

[98] It's the direct sum of the trivial representation and the $(n^2 - 1)$-dimensional adjoint representation (Eichmann (2020), appendix A.3).

[99] The adjoint representation is faithful for $SU(n)/Z$ but not for $SU(n)$.

20 Example: $SU(6)/\mathbb{Z}_3$

The previous section considered the quotient of $SU(n)$ by its full center. If the integer n is not prime, then we can also consider quotients of $SU(n)$ by a proper subgroup of its center, like $SU(6)/\mathbb{Z}_3$, where $\mathbb{Z}_3$ is understood as an abbreviation for the subgroup consisting of multiples of the identity matrix with a coefficient z satisfying $z^3 = 1$.

If ρ is the defining (hence faithful) representation of $SU(6)$, then $\rho \otimes \rho \otimes \rho$ a non-faithful representation of $SU(6)$,[100] but it is faithful as a representation of $SU(6)/\mathbb{Z}_3$.[101] The intuition is similar to the intuition used in section 19.

[100]In this representation, the matrix that represents $g \in SU(6)$ is $\rho(g) \otimes \rho(g) \otimes \rho(g)$.

[101]It's not irreducible, but that's not required here.

21 Example: $(SU(2)\times SU(2))/\mathbb{Z}_2$

This section constructs faithful representations of two groups of the form

$$\frac{SU(2)\times SU(2)}{\Gamma}, \tag{18}$$

namely the groups resulting from these two choices for the denominator:[102]

$$\Gamma = \big\{(1,1),\,(-1,1)\big\} \tag{19}$$

$$\Gamma = \big\{(1,1),\,(-1,-1)\big\}. \tag{20}$$

Both of these Γs are subgroups of $SU(2)\times SU(2)$, and both are isomorphic to $\mathbb{Z}_2$.

First consider the case (19). With this Γ, the group (18) is the same as

$$\frac{SU(2)}{\mathbb{Z}_2}\times SU(2)\simeq SO(3)\times SU(2), \tag{21}$$

where $\mathbb{Z}_2$ denotes the center of $SU(2)$. Section 19 showed how to construct a faithful representation of $SU(2)/\mathbb{Z}_2$. If ρ_3 and ρ_2 are faithful representations of $SU(2)/\mathbb{Z}_2$ and $SU(2)$, respectively, then

$$\rho_{\text{sum}}(g,g')\equiv\rho_3(g)\oplus\rho_2(g') \tag{22}$$

defines a faithful representation ρ_{sum} of the quotient group (21). This is faithful because (21) is a direct product (section 7). Another faithful representation ρ_{prod} of the same group (21) is defined by

$$\rho_{\text{prod}}(g,g')\equiv\rho_3(g)\otimes\rho_2(g'). \tag{23}$$

This is faithful because the center of $SU(2)/\mathbb{Z}_2$ is trivial, so any scalar factor $z\neq 1$ can be unambiguously associated with the second factor in (21).

[102] "1" denotes the identity matrix of whatever size is appropriate for the context (section 4).

Now consider the case (20). With this Γ, the group (18) is[103]

$$\frac{SU(2)\times SU(2)}{\Gamma}\simeq SO(4). \tag{24}$$

If ρ_2 is a faithful representation of $SU(2)$, then

$$\rho(g,g')\equiv\rho_2(g)\otimes\rho_2(g') \tag{25}$$

defines a faithful representation ρ of the quotient group (24). The intuition is similar to the intuition used in section 19.

The two quotient groups (21) and (24) both have the form (18) with $\Gamma\simeq\mathbb{Z}_2$, and they are homeomorphic to each other as topological spaces,[103] but this doesn't imply that they are isomorphic to each other as Lie groups. The smallest faithful representations of $SO(3)$ and $SU(2)$ are 3- and 2-dimensional, respectively. Using these as ρ_3 and ρ_2, the right-hand side of the isomorphism (21) shows that the representations (22) and (23) are 5-dimensional and 6-dimensional, respectively,[104] and the right-hand side of the isomorphism (24) shows that the representation (25) is 4-dimensional. The representations (22) and (25) are presumably[105] the smallest faithful representations of the quotient groups (21) and (24), respectively. Two groups whose smallest faithful representations have different sizes cannot be isomorphic to each other.

---

[103] Article 92035

[104] As representations of the group (21), (23) is irreducible and (22) is not. Both are faithful.

[105] I don't have a proof, but it seems clear intuitively.

22 Example: $(SU(n) \times SU(n))/\mathbb{Z}_n$

This section constructs faithful representations of two groups of the form

$$\frac{SU(n) \times SU(n)}{\Gamma}, \tag{26}$$

namely the groups resulting from these choices for the denominator:

$$\Gamma = \big\{(z^*, z) \mid z^n = 1\big\} \tag{27}$$

$$\Gamma = \big\{(z, z) \mid z^n = 1\big\}. \tag{28}$$

Each of these Γs is isomorphic to $\mathbb{Z}_n$. If ρ_n is the defining representation of $SU(n)$ and ρ_n^* is its dual, then the representations ρ and $\tilde{\rho}$ defined by

$$\begin{aligned} \rho(g, g') &\equiv \rho_n(g) \otimes \rho_n(g') \\ \tilde{\rho}(g, g') &\equiv \rho_n(g) \otimes \rho_n^*(g') \end{aligned}$$

are faithful representations of the groups (26) with (27) and (28) in the denominator, respectively.

When $n = 2$, the two choices for Γ shown above are equal to each other. This is consistent with the fact that the defining representation ρ_2 of $SU(2)$ and its dual ρ_2^* are equivalent to each other. To show that they're equivalent, choose a basis in which ρ_2 consists of matrixes of the form

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where a, b are complex numbers with $|a|^2 + |b|^2 = 1$. Then $\rho_2^* = Y^{-1}\rho_2 Y$ with

$$Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

23 Example: $(SU(2)\times U(1))/\mathbb{Z}_2$

This section constructs faithful representations of three groups of the form

$$\frac{SU(2)\times U(1)}{\Gamma}, \tag{29}$$

namely the groups resulting from these choices for the denominator:[106]

$$\Gamma = \big\{(1,1),\,(-1,1)\big\} \qquad \Gamma = \big\{(1,1),\,(1,-1)\big\} \qquad \Gamma = \big\{(1,1),\,(-1,-1)\big\}.$$

All three of these Γs are isomorphic to $\mathbb{Z}_2$. The first two choices for Γ give

$$\frac{SU(2)}{\mathbb{Z}_2}\times U(1) \simeq SO(3)\times U(1) \tag{30}$$

$$SU(2)\times\frac{U(1)}{\mathbb{Z}_2} \simeq SU(2)\times U(1), \tag{31}$$

respectively. These are both product groups, so we can get a faithful representation of each one by taking the direct sum of faithful representations of the factors (section 7). The third choice for Γ gives[107]

$$\frac{SU(2)\times U(1)}{\Gamma} \simeq U(2). \tag{32}$$

If ρ_2 and ρ_1 are faithful representations of $SU(2)$ and $U(1)$, then $\rho_2\otimes\rho_1$ is a representation of (32) because it doesn't distinguish between $(1,1)$ and $(-1,-1)$, and it's faithful because those are the only elements of the group $SU(2)\times U(1)$ that it maps to the identity.[108]

[106] With these three choices for Γ, the corresponding three quotient groups (30), (31), and (32) are all non-isomorphic. The groups (30) and (31) cannot be isomorphic to each other because their centers have different topologies (namely S^1 and $\mathbb{Z}_2\times S^1$, respectively). Neither (30) nor (31) has any 2-dimensional faithful representations, but the group (32) does, so neither of them is isomorphic to (32).

[107] Article 92035

[108] The defining representation of $U(2)$ is faithful, of course, but describing it as $\rho_2\otimes\rho_1$ is useful for constructing an action with different coupling constants for the two factors.

24 Example: $(SU(n) \times U(1))/\mathbb{Z}_n$

This section constructs faithful representations of four groups of the form

$$\frac{SU(n) \times U(1)}{\Gamma}, \tag{33}$$

namely the groups resulting from these choices for the denominator:

$$\Gamma = \big\{(z, 1) \mid z^n = 1\big\} \tag{34}$$
$$\Gamma = \big\{(1, z) \mid z^n = 1\big\} \tag{35}$$
$$\Gamma = \big\{(z^*, z) \mid z^n = 1\big\} \tag{36}$$
$$\Gamma = \big\{(z, z) \mid z^n = 1\big\}. \tag{37}$$

Each of these Γs is isomorphic to $\mathbb{Z}_n$. The resulting quotient groups are[109]

$$\frac{SU(n)}{\mathbb{Z}_n} \times U(1) \tag{38}$$
$$SU(n) \times \frac{U(1)}{\mathbb{Z}_n} \simeq SU(n) \times U(1) \tag{39}$$
$$\frac{SU(n) \times U(1)}{\big\{(z^*, z) \mid z^n = 1\big\}} \simeq U(n) \tag{40}$$
$$\frac{SU(n) \times U(1)}{\big\{(z, z) \mid z^n = 1\big\}}. \tag{41}$$

The groups (38) and (39) are both product groups, so we can get a faithful representation of each one by taking the direct sum of faithful representations of the factors.[110] If ρ_n and ρ_1 are faithful representations of $SU(n)$ and $U(1)$, then $\rho_n \otimes \rho_1$ is a representation of (40) because it does not distinguish between elements (z^*, z) with $z^n = 1$, and it is faithful because those are the only elements of the group $SU(n) \times U(1)$ that it maps to the identity. Similarly, $\rho_n^* \otimes \rho_1$ is a faithful representation of (41), where ρ_n^* is the dual of ρ_n.

[109] Article 92035 cites a reference for the isomorphism (40).

[110] Section 19 constructed a faithful representation of $SU(n)/\mathbb{Z}_n$.

25 Example: $(SU(2)\times SU(6))/\mathbb{Z}_2$

Now consider groups of the form

$$\frac{SU(2)\times SU(6)}{\Gamma} \tag{42}$$

with any of these choices for the denominator:

$$\begin{aligned}
\Gamma &= \big\{(1,1),\,(-1,1)\big\}\\
\Gamma &= \big\{(1,1),\,(1,-1)\big\}\\
\Gamma &= \big\{(1,1),\,(-1,-1)\big\}.
\end{aligned}$$

For the first Γs, the quotient group (42) is

$$\frac{SU(2)}{\mathbb{Z}_2}\times SU(6) \qquad\qquad SU(2)\times\frac{SU(6)}{\mathbb{Z}_2}, \tag{43}$$

respectively, where $\mathbb{Z}_2$ is an abbreviation for the two-element subgroup of the center of the respective numerator. The preceding sections illustrated how to construct faithful representations of groups of the form $SU(n)/\mathbb{Z}_k$ with $\mathbb{Z}_k\subset Z(SU(n))$. In each of the cases (43), elements of the (remaining) centers of the two factors can't cancel each other,[111] so the tensor product of faithful representations of the factors gives a faithful representation of the product group.

Now consider the third choice for Γ. In that case, if ρ_2 and ρ_6 are the defining representations of $SU(2)$ and $SU(6)$, respectively, then $\rho_2\otimes\rho_6$ is a faithful representation of the quotient group (42). The intuition is similar to the intuition used in section 19.

[111] In the first case, this is true because the center of the first factor is trivial. In the second case, this is true because $z=1$ is the only complex number satisfying both $z^2=1$ and $z^3=1$.

26 Example: $(SU(6)\times SU(6))/\mathbb{Z}_3$

Now consider the group

$$\frac{SU(6)\times SU(6)}{\Gamma} \tag{44}$$

where Γ is the three-element sugroup

$$\Gamma = \big\{(z, z^*) \mid z^3 = 1\big\}. \tag{45}$$

Let ρ_6 be the defining representation of $SU(6)$, and let ρ' be the faithful representation of $SU(6)/\mathbb{Z}_3$ that was constructed in section 20. Then

$$\rho(g_1, g_2) = \big(\rho_6(g_1)\otimes\rho_6(g_2)\big)\oplus\rho'(g_1)\oplus\rho'(g_2) \tag{46}$$

is a faithful representation of (44) when Γ is given by (45). Intuition:

- This representation's kernel clearly contains Γ.
- A pair (g_1, g_2) cannot be in the representation's kernel unless it's in the kernel of each term in the direct sum.
- A pair (g_1, g_2) cannot be in the kernel of $\rho_6(g_1)\otimes\rho_6(g_2)$ unless g_1 and g_2 are each other's inverses and are in the center of $SU(6)$.
- A pair (g_1, g_2) cannot be in the kernels of $\rho'(g_1)$ and $\rho'(g_2)$ unless g_1 and g_2 are both in the $\mathbb{Z}_3$ subgroup of the center of $SU(6)$.

Altogether, this implies that the kernel of the representation (46) is equal to (45), so this is a faithful representation of (44).

27 Example: $(U(1) \times U(1))/\mathbb{Z}_n$

This section constructs faithful representations of groups of the form

$$\frac{U(1) \times U(1)}{\Gamma}, \tag{47}$$

with either of these choices for Γ:

$$\Gamma = \big\{(z^*, z) \mid z^n = 1\big\} \tag{48}$$
$$\Gamma = \big\{(z, z) \mid z^n = 1\big\}. \tag{49}$$

We can think of $U(1) \times U(1)$ as the group of matrixes of the form

$$(e^{i\theta},\, e^{i\phi}) \equiv \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

for all $\theta, \phi \in \mathbb{R}$. We can construct a representation of (47) by giving a homomorphism from $U(1) \times U(1)$ to itself whose kernel is precisely Γ. The homomorphism defined by

$$(e^{i\theta},\, e^{i\phi}) \to (e^{in\theta}, e^{i(\phi+\theta)}) \tag{50}$$

has this property when Γ is given by (48), and the homomorphism defined by

$$(e^{i\theta},\, e^{i\phi}) \to (e^{in\theta}, e^{i(\phi-\theta)}) \tag{51}$$

has this property when Γ is given by (49). These show that the quotient groups (47) are isomorphic to $U(1) \times U(1)$ itself, even though they are both realized as proper subgroups of the original $U(1) \times U(1)$ when $n > 1$.

When $n = 1$, the homomorphisms (50) and (51) are each other's inverses: their composition is the identity transformation. This shows that the $n = 1$ versions of (50) and (51) are automorphisms of $U(1) \times U(1)$ (isomorphisms with itself), even though they don't preserve the original product structure.

28 Example: $(SU(3) \times SU(2) \times U(1) \times U(1))/\Gamma$

Now consider the group

$$\frac{SU(3) \times SU(2) \times U(1) \times U(1)}{\Gamma} \tag{52}$$

with

$$\Gamma = \big\{(a^*, b^*, az^*, bz) \mid a^3 = 1,\ b^2 = 1,\ z^5 = 1\big\}. \tag{53}$$

Let ρ_3, ρ_2, ρ_1 be the defining representations of $SU(3)$, $SU(2)$, and $U(1)$. Then the representation of (52) defined by[112]

$$\begin{aligned}\rho(g, h, z, z') = &\big(\rho_3(g) \otimes \rho_2(h) \otimes \rho_1(z) \otimes \rho_1(z')\big)\\ &\oplus \Big(\big(\rho_3(g)\big)^5 \otimes \big(\rho_1(z)\big)^5\Big)\\ &\oplus \Big(\big(\rho_2(h)\big)^5 \otimes \big(\rho_1(z')\big)^5\Big)\end{aligned} \tag{54}$$

is faithful. Intuition:

- This representation's kernel clearly contains Γ.
- (g, h, z, z') cannot be in the representation's kernel unless it's in the kernel of each term in the direct sum.
- (g, h, z, z') cannot be in the kernel of the first term in the direct sum unless $g \in Z(SU(3))$, $h \in Z(SU(2))$, and $ghzz' = 1$.[113]
- (g, h, z, z') cannot be in the kernels of the second and third terms in the direct sum unless unless $g \in Z(SU(3))$, $h \in Z(SU(2))$, $(gz)^5 = 1$, and $(hz')^5 = 1$.

Altogether, this implies that the kernel of the representation (54) is equal to (53), so this is a faithful representation of (52).

[112]Equation (54) uses the abbreviation M^5 for the tensor product of 5 copies of M.

[113]I'm using the same symbol for an element of the center and the overall complex coefficient of the otherwise-identity matrix that represents that element of the center (section 8).

29 References

Aharony *et al*, 2013. "Reading between the lines of four-dimensional gauge theories" *Journal of High Energy Physics* **2013**: 115, https://arxiv.org/abs/1305.0318

Babu *et al*, 1998. "Implications of Generalized Z-Z' Mixing" *Phys. Rev. D* **57**: 6788, https://arxiv.org/abs/hep-ph/9710441

Baez and Huerta, 2010. "The Algebra of Grand Unified Theories" *Bull. Am. Math. Soc.* **47**: 483-552, https://arxiv.org/abs/0904.1556

Creutz and Moriarty, 1982. "Phase Transitions in U(2) and U(3) Lattice Gauge Theory in Four-dimensions" *Phys. Rev. D* **25**: 610, https://lib-extopc.kek.jp/preprints/PDF/1981/8110/8110258.pdf

Davighi *et al*, 2020. "Global anomalies in the Standard Model(s) and beyond" *JHEP* **07**: 232, https://link.springer.com/article/10.1007/JHEP07(2020)232

De Cesare *et al*, 2021. "Five-dimensional CFTs from the ϵ-expansion" *Physical Review D* **104**: 105015, https://arxiv.org/abs/2107.00342

Eichmann, 2020. "Appendix A: $SU(N)$" http://cftp.ist.utl.pt/~gernot.eichmann/lecture-notes

Emel'yanov and Petrovskiĭ, 1983. "Topological plroperties of a one-plaquette action and phase transitions in a lattice gauge theory" *JETP Lett.* **38**: 302-304, http://jetpletters.ru/ps/1822/article_27858.pdf

Florio *et al*, 2021. "Searching for continuous phase transitions in 5D SU(2) lattice gauge theory" *JHEP* **12**: 076, https://link.springer.com/article/10.1007/JHEP12(2021)076

Fulton and Harris, 1991. *Representation Theory: A First Course.* Springer

Gotô, 1949. "A Theorem on compact semi-simple groups" *Journal of the Mathematical Society of Japan* **1**: 270-272, https://projecteuclid.org/journals/jmsj/volume-1/issue-3/A-Theorem-on-compact-semi-simple-groups/10.2969/jmsj/00130270.full

Hall, 2015. *Lie Groups, Lie Algebras, and Representations (Second Edition).* Springer

Harlow and Ooguri, 2021. "Symmetries in Quantum Field Theory and Quantum Gravity" *Communications in Mathematical Physics* **383**: 1669-1804, https://arxiv.org/abs/1810.05338

Jaeckel and Ringwald, 2010. "The Low-Energy Frontier of Particle Physics" *Ann. Rev. Nucl. Part. Sci.* **60**: 405-437, https://arxiv.org/abs/1002.0329

Kirillov and Kirillov, 2005. "Compact groups and their representations" https://arxiv.org/abs/math/0506118v1

Knapp, 2023. *Lie Groups Beyond an Introduction.* Anthony W. Knapp, https://www.math.stonybrook.edu/~aknapp/download/Beyond2.pdf

Kramer, 2023. "A note on commutators in compact semisimple Lie algebras" *Innov. Incidence Geom.* **20**: 317-324, https://arxiv.org/abs/2204.09913

Lohitsiri, 2020. "Anomalies and the Standard Model of Particle Physics" https://www.repository.cam.ac.uk/items/8c631fba-e456-4410-a387-3c72915e12c4

Luo and Xiao, 2003. "Renormalization Group Equations in Gauge Theories with Multiple U(1) Groups" *Phys. Lett. B* **555**: 279-286, https://arxiv.org/abs/hep-ph/0212152

Malkoun and Nahlus, 2017. "Commutators and Cartan Subalgebras in Lie Algebras of Compact Semisimple Lie Groups" *Journal of Lie Theory* **27**: 1027-1032, https://arxiv.org/abs/1602.03479

Morel, 2019. "Representation theory" https://web.math.princeton.edu/~smorel/rep_theory_notes.pdf

Sepanski, 2007. *Compact Lie Groups.* Springer

Teper, 2018. "SO(4), SO(3) and SU(2) gauge theories in 2+1 dimensions: comparing glueball spectra and string tensions" https://arxiv.org/abs/1801.05693

Tong, 2017. "Line Operators in the Standard Model" *JHEP* **07**: 104, https://link.springer.com/article/10.1007/JHEP07(2017)104

Tong, 2018. "Gauge theory" https://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

30 References in this series

Article **07611** (https://cphysics.org/article/07611):
"Asymptotic Freedom and the Continuum Limit of Yang-Mills Theory" (version 2024-10-18)

Article **10142** (https://cphysics.org/article/10142):
"Universality and Continuum Limits with Scalar Quantum Fields" (version 2024-10-18)

Article **26542** (https://cphysics.org/article/26542):
"The Free Quantum Electromagnetic Field in Smooth Space" (version 2024-03-04)

Article **28539** (https://cphysics.org/article/28539):
"Homology Groups" (version 2024-11-10)

Article **29682** (https://cphysics.org/article/29682):
"A Quick Review of Group Theory: Homomorphisms, Quotient Groups, Representations, and Extensions" (version 2024-05-21)

Article **61813** (https://cphysics.org/article/61813):
"Homotopy, Homotopy Groups, and Covering Spaces" (version 2024-05-19)

Article **89053** (https://cphysics.org/article/89053):
"Quantum Gauge Fields on a Spacetime Lattice" (version 2024-11-10)

Article **91563** (https://cphysics.org/article/91563):
"Characterizing the Irreducible Representations of Compact Simple Lie Groups" (version 2024-11-30)

Article **92035** (https://cphysics.org/article/92035):
"The Topology of Lie Groups: a Collection of Results" (version 2024-11-27)