A Quick Review of Differential Forms and Stokes's Theorem

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Abstract This article very briefly reviews some basic definitions and facts about differential forms that are used by other articles in this series, including Stokes's theorem and the definition of the Hodge dual (also called the Hodge star).

Contents

1 Introduction

To motivate the subject of this article, I'll quote the first page of Tao (2020), slightly reformatted:

> The concept of integration is of course fundamental in single-variable calculus. Actually, there are *three* concepts of integration which appear in the subject:
>
> - the indefinite integral...
> - the unsigned definite integral...
> - the signed definite integral...
>
> These three integration concepts are of course closely related to each other in single-variable calculus... When one moves from single-variable calculus to several-variable calculus, though, these three concepts begin to diverge significantly from each other. ...the signed definite integral generalises to the integration of forms...

The last sentence in the quote is referring to **differential forms**, the subject of this article. This article briefly reviews some basic definitions and facts about differential forms that are used by other articles in this series.[1]

Section 9 is an exception: it reviews a concept that generalizes the unsigned definite integral instead.

[1] Chapters 14-16 in Lee (2013) introduce the subject more completely.

2 Differential forms

Differential forms live on smooth manifolds[2] that may have boundaries.[3] For any integer $m \geq 1$, an ***m*-form** is a map that takes m vector fields as input, returns a single scalar field as output,[4] is linear in each of its inputs, and is antisymmetric with respect to permutations of its inputs.[5] Special cases:

- A **zero-form** is just a scalar field (an ordinary function whose domain is the manifold).
- A **one-form** is a linear map whose input is a vector field and whose output is a scalar field.
- A **two-form** ω is an antisymmetric linear map whose input is a pair of vector fields v_1, v_2 whose output $\omega(v_1, v_2)$ is a scalar field. *Antisymmetric* means $\omega(v_1, v_2) = -\omega(v_2, v_1)$.

When m doesn't need to be specified, an m-form is also called a **differential form**. The integer m is called the **degree** of the differential form.

A linear combination of m-forms is still an m-form, even if the coefficients are functions: if $\omega_1, \omega_2, ...$ are all m-forms (with the same degree m) and $f_1, f_2, ...$ are ordinary real- or complex-valued functions on the manifold, then $\sum_k f_k\omega_k$ is also an m-form.

[2]Article 93875 reviews the concept of a **smooth mnanifold** without boundary.

[3]Article 44113 extends the definition of *smooth manifold* to include manifolds with boundaries.

[4]In this article, a **scalar field** is a map from the manifold to either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers). This leads to ordinary differential forms. Differential forms with coefficients in other structures can also be defined, as described in the section titled *twisted and vector-valued forms* in `https://ncatlab.org/nlab/show/differential+form` (2025-01-26).

[5]Article 09894

3 Exterior product

The **exterior product** (or **wedge product**) of an m-form and an m'-form is an $(m+m')$-form. Instead of reviewing the complete definition, here are a few of its key properties:[6] it is associative, linear in each factor, and anticommutative. **Anticommutative** means

$$\alpha \wedge \beta = (-1)^{ab}\beta \wedge \alpha$$

whenever α is an a-form and β is a b-form. Example: if $\alpha_1, ..., \alpha_N$ are one-forms, then the N-form

$$\omega \equiv \alpha_1 \wedge \cdots \wedge \alpha_N \tag{1}$$

is antisymmetric under permutations of the factors α_n.[7] This implies

$$\omega(v_1, ..., v_N) = \kappa \sum_\pi (-1)^\pi \alpha_1(v_{\pi(1)}) \cdots \alpha_N(v_{\pi(N)})$$

for every N-tuple of vector fields $v_1, ..., v_N$. The sum is over all permutations, the factor $(-1)^\pi$ is defined to be 1 for even permutations and -1 for odd permutations, and κ is a normalization factor. The normalization convention $\kappa = 1$ is standard.[8]

In this article, a differential form will be called **decomposible** if it can be written as an exterior product of one-forms,[9] like in equation (1).

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[6] Lee (2013), proposition 14.11 combined with the text above equation (14.14)

[7] Article 81674 uses this product with vectors in place of one-forms. This gives a nice way to define (and to think about) the determinant of a linear transformation.

[8] Nakahara (1990), equation (5.62)

[9] The text above exercise 14.12 in Lee (2013) introduces this word in a more limited context.

4 Orientation

Some smooth manifolds are orientable, and some are not. Examples of orientable manifolds include spheres and tori. Examples of non-orientable manifolds include even-dimensional real projective spaces.[10,11]

One way to define this is by using top-degree differental forms.[12] In the context of an n-dimensional manifold $\mathcal{M}$, and n-form is also called a **top-degree form**. A manifold $\mathcal{M}$ is **orientable** if it admits a top-degree form ω that is not zero anywhere, also called an **orientation form**.[13]

Two orientation forms ω and ω' on $\mathcal{M}$ define the same **orientation** of $\mathcal{M}$ if $\omega' = f\omega$ for some function f that is positive everywhere on $\mathcal{M}$. An orientation form with the opposite sign, $-\omega$, defines the **opposite orientation** for $\mathcal{M}$, where *opposite* means compared to the orientation defined by ω. An orientable manifold admits exactly two possible orientations. An orientable manifold together with a choice of one of its two possible orientations is called an **oriented manifold**.

If ω is an orientation form on an n-dimensional manifold $\mathcal{M}$ and β is an orientation form on its boundary $\partial\mathcal{M}$, then the orientation defined by β is said to be **induced** by the orientation defined by ω if they are related to each other by[14]

$$\omega(w, v_1, v_2, ..., v_n) = \beta(v_1, v_2, ..., v_n)$$

for some vector field w on $\mathcal{M}$ that is outward-pointing everywhere on $\partial\mathcal{M}$, for all vector fields $v_1, ..., v_n$ on $\mathcal{M}$.[15] This convention is used to select one of the two possible orientations of $\partial\mathcal{M}$ based on a given orientation of $\mathcal{M}$.

[10] Article 28539

[11] Odd-dimensional real projective spaces are orientable (article 28539).

[12] Lee (2013), proposition 15.5

[13] It is also called a **volume form**, but that name is often reserved for a specific orientation form defined with the help of a designated metric tensor (section 10 and `https://ncatlab.org/nlab/show/volume+form`), and the same name is also sometimes used for a **volume pseudoform** or **volume density** that does not require the manifold to be oriented (section 9).

[14] Lee (2013), text above lemma 14.13, proposition 15.24, and its proof

[15] Recall that an m-form is a special kind of map whose input is a list of m vector fields and whose output is a scalar field (section 2).

5 Exterior derivatives

Each point in a manifold has a neighborhood whose points can be parameterized using a single coordinate system. This is called a **coordinate chart** for that neighborhood. Within one coordinate chart, we can think of each coordinate x_k as a function from that part of the manifold to $\mathbb{R}$. The **differential** dx_k of x_k is a one-form defined on that part of the manifold by

$$dx_k(\partial_j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where ∂_j denotes the partial derivative with respect to x_j. Each of these partial derivatives is a vector field,[16] and every vector field in that chart may be written as a linear combination of these with functions as coefficients, so (2) defines the one-form dx_k completely (within that chart).

The **exterior derivative** is a map from m-forms ω to $(m+1)$-forms $d\omega$ with these properties (among others):[17]

- If $f(x_1, x_2, ...)$ is a zero-form (an ordinary function), then $df = \sum_k (\partial f/\partial x_k)\, dx_k$.
- If $\beta = d\alpha$, then $d\beta = 0$.
- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^a \alpha \wedge d\beta$ whenever α is an a-form.

A differential form ω is called **closed** if $d\omega = 0$, and it's called **exact** if it can be written $\omega = d\alpha$ for some other differential form α. The second property listed above says that every exact differential form is closed. The converse is not always true – a closed differential form might not be exact[18] – but a weaker statement is true: each point of a smooth manifold has a neighborhood in which every closed form is exact.[19]

[16] Article 09894

[17] Lee (2013), theorem 14.24

[18] This is the foundation for **de Rham cohomology**, which uses the existence of non-exact closed differential forms to make statements about the topology of the smooth manifold that hosts them (Lee (2013), chapter 17).

[19] Lee (2013), corollary 17.15

6 Integration on orientable manifolds

An m-form defined on an n-dimensional manifold $\mathcal{M}$ can be integrated over an m-dimensional submanifold $\mathcal{S} \subset \mathcal{M}$ ($m \leq n$). If the topology of $\mathcal{S}$ is trivial so that it can be covered by a single coordinate chart, and if the m-form is

$$\omega \equiv f(x_1, ..., x_m)\, dx_1 \wedge \cdots \wedge dx_m,$$

then the definition is simply

$$\int_{\mathcal{S}} \omega \equiv \int f(x_1, ..., x_m)\, dx_1 \cdots dx_m. \tag{3}$$

The integral on the right side is defined as usual. Linearity can be used to extended this definition to arbitrary m-forms, and chapter 16 in Lee (2013) explains how to extend it to manifolds that cannot be covered by a single contractible chart.

When an integrand originally expressed in one coordinate system is re-expressed in another one, the jacobian factor associated with that change of variables comes automatically from properties (like antisymmetry) of the exterior derivative and exterior product.[20] For that reason, bundling the integrand as an m-form ω, like on the left side of (3), makes "the integral of ω over $\mathcal{M}$" unambiguous without needing to specify a coordinate system. This works because ω automatically includes the jacobian factor.[21,22] For the same reason, the integral is invariant under orientation-preserving diffeomorphisms.[23,24]

[20] Taylor (2006), appendix G and appendix H

[21] Ivanov (2001) uses a relatively easy-to-prove special case of Stokes's theorem to derive the change-of-variables formula for ordinary multivariable integrals.

[22] Here, the **jacobian factor** is a signed determinant, not its absolute value, in contrast to the definition that section 9 will review. (This distinction is also mentioned in `https://ncatlab.org/nlab/show/pseudoform`.)

[23] Lee (2013), first page in chapter 16

[24] Whether the condition *orientation-preserving* is needed depends on what the diffeomorphism is applied to. Consider the ordinary integral $\int_\infty^\infty e^{-x^2}\, dx$. We can write this as $\int_{\mathcal{I}} \omega$ where $\mathcal{I}$ is the one-dimensional manifold $\mathbb{R}$ with the appropriate orientation and ω is the one-form $e^{-x^2}\, dx$. If we apply an orientation-reversing diffeomorphism to ω but not to $\mathcal{I}$, then the integral changes sign. If we apply that orientation-reversing diffeomorphism to both ω and $\mathcal{I}$, then it doesn't. The concept of a *change of integration variables* applies the transformation to both ω and $\mathcal{I}$.

7 Stokes's theorem

Let ω be an $(n-1)$-form defined on an n-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. **Stokes's theorem**[25] relates the integral of $d\omega$ over $\mathcal{M}$ to the integral of ω over $\partial\mathcal{M}$. The theorem says[26,27] that if ω has compact support, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega. \tag{4}$$

Section 8 will demonstrate that the *compact support* condition is essential.

Sometimes the theorem is stated in the context of a larger n-dimensional manifold that includes $\mathcal{M}$ as an oriented n-dimensional submanifold,[28] and sometimes the condition for ω to have compact support is implicitly enforced by assuming that $\mathcal{M}$ itself is compact.[29]

Equation (4) clearly assumes a relationship between the orientations of $\mathcal{M}$ and $\partial\mathcal{M}$, because reversing the orientation changes the sign of the integral.[30] The assumed orientation of $\partial\mathcal{M}$ is the one *induced* from the orientation of $\mathcal{M}$, as defined in section 4. To check that this is consistent with the relative signs in (4), use the standard coordinate system $(x_1, ..., x_n)$ for $\mathbb{R}^n$, let $\mathcal{M}$ be the n-dimensional manifold defined by the condition $x_1 \leq 0$, and let

$$\omega = f(x_1, ..., x_n)\, dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n.$$

Then $\int_{\partial\mathcal{M}} \omega = \int f(0, x_2, ..., x_n)\, dx_2\, dx_3 ..., dx_4$ and $d\omega = (\partial f/\partial x_1)\, dx_1 \wedge \cdots \wedge dx_n$. The one-form dx_1 satisfies $dx_1(v) > 0$ for all outward-pointing (pointing in the $+x_1$ direction) vector fields on $x_1 = 0$, so the relative signs in (4) are consistent with the definition of *induced orientation* in section 4.

[25] https://www.englishrules.com/writing/2005/possessive-form-of-singular-nouns-ending-with-s/

[26] Lee (2013), theorem 16.11

[27] Theorem 16.25 in Lee (2013) generalizes this to manifolds with *corners* (article 44113).

[28] Madsen and Tornehave (1997), theorem 10.8

[29] Eliashberg (2018), theorem 11.1

[30] Lee (2013), proposition 16.6(b)

8 The importance of compact support

Here's an example demonstrating that the *compact support* condition is essential in Stokes's theorem. Let (x, y) be the standard coordinate system for $\mathbb{R}^2$, and let $\mathcal{M}$ be the submanifold of $\mathbb{R}^2$ defined by $0 < r \leq 1$ with $r \equiv \sqrt{x^2 + y^2}$. The one-form

$$\omega \equiv \frac{x\, dy - y\, dx}{r^2} \tag{5}$$

is defined everywhere on $\mathcal{M}$. The next two paragraphs will derive the results

$$\int_{\partial\mathcal{M}} \omega = 2\pi \tag{6}$$

and

$$\int_{\mathcal{M}} d\omega = 0. \tag{7}$$

This evades Stokes's theorem because ω does not have compact support. This is clear from the fact that ω is nonzero everywhere on $\mathcal{M}$ combined with the fact that $\mathcal{M}$ is a non-compact manifold.[31]

To deduce (7), use the identity $d\omega = f\, d\alpha + df \wedge \alpha$ with $f = r^{-2}$ and $\alpha = x\, dy - y\, dx$ to get $d\omega = 0$.

To deduce (6), define ϕ by the conditions $x + iy = re^{i\phi}$ and $0 < \phi < 2\pi$. The function ϕ is defined almost everywhere on $\mathcal{M}$, but not on the line segment defined by $y = 0$ and $x > 0$. The one-form (5) may be written as $\omega = d\phi$ everywhere on the circle $\partial\mathcal{M}$ except the one point $(x, y) = (1, 0)$, where ϕ is undefined. This gives (6) because (5) defines a smooth one-form everywhere on the circle, and omitting a single point does not change the value of the integral.

This shows that the one-form ω is closed but not exact.[32] It can be written $\omega = d\phi$ for a (single-valued) function that is defined *almost* everywhere on $\mathcal{M}$, but it cannot be written as $\omega = df$ for any (single-valued) smooth function f defined on all of $\mathcal{M}$.

[31] It would be compact if the point $r = 0$ were included.

[32] Section 5

9 Integration on non-orientable manifolds

As section 6 reviewed, m-forms can be integrated over *orientable* m-dimensional manifolds. This works because if ω is an m-form, then[33]

$$\omega(Lv_1, ..., Lv_m) = (\det L)\, \omega(v_1, ..., v_m) \tag{8}$$

for every m-tuple of vector fields $v_1, ..., v_m$ and every linear transformation L. The determinant accounts for the (signed) jacobian factor.

The definition of the integral of ω doesn't work for a non-orientable manifold $\mathcal{M}$ because then the integrals of ω over charts that cover $\mathcal{M}$ cannot be stitched together with mutually consistent signs over all of $\mathcal{M}$. A **density** is a different kind of object that can be integrated over a non-orientable manifold $\mathcal{M}$. For any linear transformation L, a density ω satisfies[34]

$$\omega(Lv_1, ..., Lv_m) = |\det L|\, \omega(v_1, ..., v_m)$$

instead of (8). An m-form is linear in each of its inputs, but a density is not:[35] changing the sign of one of its inputs does not change the sign of the density. Chapter 16 in Lee (2013) explains how to define the integral of a density over a not-necessarily-orientable manifold.

Every smooth m-dimensional manifold $\mathcal{M}$ admits a nowhere-zero density of degree m.[36]

[33] Article 81674

[34] Lee (2013), equation (16.18)

[35] For this reason, a density is not a tensor field (Lee (2013), text below equation (16.18)).

[36] Lee (2013), proposition 16.37

10 The Hodge dual of a differential form

The concepts reviewed in section 2-9 don't depend on a metric tensor, which is an additional piece of structure used to define angles and distances (and time intervals, if the signature is lorentzian).[37,38] A **metric tensor** g takes two vector fields as input and gives a scalar field as output, and it's symmetric: $g(v,w) = g(w,v)$. Given a metric tensor, we can define the corresponding **inverse metric tensor** $\bar{g}$ that defines an inner product between one-forms.[39] This can be extended to an inner product between m-forms for each integer m. Example: if $\alpha_1, \alpha_2, \beta_1, \beta_2$ are one-forms, then the inner product between the two-forms $\alpha \equiv \alpha_1 \wedge \alpha_2$ and $\beta \equiv \beta_1 \wedge \beta_2$ is

$$\bar{g}(\alpha, \beta) \equiv \bar{g}(\alpha_1, \beta_1)\bar{g}(\alpha_2, \beta_2) - \bar{g}(\alpha_1, \beta_2)\bar{g}(\alpha_2, \beta_1).$$

This is consistent with the antisymmetry of the exterior product. The definition for decomposable m-forms of any degree m follows the same pattern, and it can be extended to non-decomposable two-forms using linearity.

In the context of an n-dimensional smooth manifold with a designated metric tensor, the **volume form** ν is the unique top-degree form whose integral over a hypercube with edges represented by n infinitesimal orthogonal vectors v_k is $|\prod_k g(v_k, v_k)|^{1/2}$.[40] *Orthogonal* is defined using the metric tensor. The **Hodge dual** of an m-form β is the $(n-m)$-form $\star\beta$ defined by[41]

$$\alpha \wedge (\star\beta) = \bar{g}(\alpha, \beta)\nu \tag{9}$$

for all m-forms α (with the same degree m as β).

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[37] Article 48968

[38] Article 09894 calls it a **metric field**, because it's a special kind of tensor field.

[39] In a chart covered by coordinates $x_1, x_2, ...$, we can write $v = v^a \partial_a$ and $w = w^a \partial_a$ (with implied sums over the index a), and then $g(v,w) = g_{ab} v^a w^b$ (with implied sums over a, b) where g_{ab} are the components of the metric tensor. Similarly, given two one-forms $\alpha = \alpha_a\, dx^a$ and $\beta = \beta_a\, dx^b$, we have $\bar{g}(\alpha, \beta) = g^{ab}\alpha_a\beta_b$ where g^{ab} are the components of the inverse metric tensor (article 09894).

[40] Morrison (1999)

[41] Morrison (1999); Lee (2013), exercise 16-18(c); `https://ncatlab.org/nlab/show/Hodge+star+operator`

11 Examples

Suppose that the manifold is $\mathbb{R}^n$ with the standard euclidean metric g, and let $x_1, ..., x_n$ be a system of coordinates in which $g(\partial_j, \partial_k) = \delta_{jk}$. Then the volume form is

$$\nu = dx_1 \wedge \cdots \wedge dx_n,$$

and the definition (9) implies[42]

$$\begin{aligned} \star 1 &= \nu \\ \star(dx_1) &= dx_2 \wedge \cdots \wedge dx_n \\ \star(dx_2) &= (-1)^{n-1} dx_3 \wedge \cdots \wedge dx_n \wedge dx_1 \\ \star(dx_1 \wedge dx_2) &= dx_3 \wedge \cdots \wedge dx_n \\ \star\nu &= 1. \end{aligned}$$

Now consider an n-dimensional manifold with a metric of any signature, and let k be the number of negative components of the diagonalized metric tensor. Then[43,44]

$$\star(\star\omega) = (-1)^k (-1)^{m(n-m)} \omega$$

for every m-form ω. In particular, the volume form satisfies

$$\star(\star\nu) = (-1)^k \nu.$$

This is consistent with (9) because the inner product of the volume form with itself is $\bar{g}(\nu, \nu) = (-1)^k$.

[42]Section 4 in Dray (1999) illustrates this for $n = 2$.

[43]Morrison (1999)

[44]Section 4 in Dray (1999) illustrates this for a metric with $k = 1$.

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13 References in this series

Article **09894** (https://cphysics.org/article/09894):
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Article **28539** (https://cphysics.org/article/28539):
"Homology Groups"

Article **44113** (https://cphysics.org/article/44113):
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